# THE KAIRO METHOD
# AND THE MODIFIED KAIRO METHOD

„NEW ALTERNATIVES FOR SOLVING LINEAR PROGRAMMING PROBLEMS"

AND

# THE KAIRO ALGORITHM FOR MATRIX INVERSION

Dissertation
zur Erlangung des Doktorgrades der
Wirtschafts- und Sozialwissenschaftlichen Fakultät
der Georg-August-Universität zu Göttingen

vorgelegt von

Mustafa M. Gouda
aus Tala / Ägypten

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Berichterstatter: Professor Dr, J. Bloech
Mitberichterstatter: Professor Dr. H. Koltermeier

Tag der mündlichen Prüfung 6. 2. 1976

To my wife Mufida
my son Hussam
my daughter Dalia
They have suffered enough whole four years

To Father and Mother
They have taught me how to learn.

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UNIV. LEIDEN BIBL.

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## List of Notations

$A$ matrix, usually of size $m.n$

$A_{rr}$ submatrix, the basis of the current solution at the $K^{th}$ iteration. $r= 1,2,\ldots,k$.

$A_{rr}^{-1}$ the inverse of the basis at the $K^{th}$ iteration.

$A_{rs}$ submatrix, the index $s$ indicates the nonbasic variables. $s = (k+1),(k+2), \ldots,n$.

$A_{dr}$ submatrix, the index $d$ indicates the nonexhausted redundant capacities. $d = (k+1),(k+2),\ldots,m$.

$A_{ds}$ submatrix, where the pivot element $a_{ge}$ to be found (case 1)

$a_{ij}$ elements of the matrix A

$a_e$ the $e^{th}$ column of the matrix A (the pivot column)

$a_g$ the $g^{th}$ row of the matrix A (the pivot row)

$B$ matrix, usually of size $m.m$ and nonsingular

$B^{-1}$ the inverse of B

$C$ row vector, the cost coefficients, usually with n components $C =( c_1 ,c_2 , \ldots,c_n )$

$I$ identity matrix

$E$ elementary matrix

$P^o$ column vector, the constant vector, usually with m components $P^o=(P_1^o ,P_2^o ,\ldots,P_m^o )$

$X_r$ subcolumn vector, the index r indicates the real basic variables . $r=1,2,\ldots,k$.

$X_s$ subcolumn vector, the index s indicates the real nonbasic variables. $s= k+1, k+2,\ldots,n$.

$Y_{rr}$ the inverse of the basis $A_{rr}$, the same as $A_{rr}^{-1}$

$Y_{re}$ subcolumn vector represents the value of the newcoming variable $x_e$ in terms of the current basis

$Y_{de}$ subcolumn vector represents the value of the newcoming variable $x_e$ in terms of the redundant capacities d, $d=(k+1),(k+2),\ldots,m$

$Y_{ge}$ the pivot element

$f_j$ the relative cost coefficients.

$Z$ the objective function.

## 1. INTRODUCTION

"Man has long been interested in, and intrigued by, optimization problems" - the search for the optimum or the extreme value of a function. Or in other words, getting the best of something under given conditions. (1)

During the last twenty or thirty years, many new and important optimization problems have emerged in the field of economics and have received a great deal of attention. As a class, these problems may be called programming problems. (2)

Broadly speaking, programming problems mean the selection of economic activities for the sake of optimization of the limited resources such as men, materials, machines, and land. (3)

Such problems are called allocation problems as well and arise when:

a) There are a number of activities to be performed and there are many ways to do them, and -

b) Resources or facilities are not available enough to utilize each capacity in the most effective way. (4)

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1) Ackoff, R. L., (ed.), "Progress in Operations Research", Vol. 1, New York, London, John Wiley & Sons, Inc. 1961, p. 108

2) Hadley, G.,"Linear Programming", Addison-Wesley Publishing Company, Inc., Reading, Mass., 1962, p. 1

3) Saaty, T. L.,"Mathematical Methods of OR", New York, MacGraw-Hill 1959, p. 165

4) Ackoff, R. L., , op. cit. p. 109

Management's primary function is to make decisions to determine the future course of action for the organization over the short and the long run. (1)

The manager of any business firm faces at least three important classes of decisions, they are:

a) how to produce?
b) how much to produce? - and if more than one product is involved,
c) what combination of products to produce?

The first class of decision involves the problem of selecting the combination of men and equipment which will produce a given quantity of output at least cost. The second class of decisions has to do with the choice of the most profitable quantity of output to produce and sell. Inherent of the output decision is the third class of decisions, what combination of products should be produced and sold? (2)

Taking such decisions, objectives such as profits, costs, quantities produced, or other measures of effectiveness are to be obtained in the best possible, or optimal fashion subject to certain restraining conditions. (3) Such decisions may be based on judgement and empirical experience. As the choices to be made become more numerous and the decisions more complicated, management has tended to look for methods to help in making decisions - and the role of judgement, and it has found in mathematics a powerful tool.

---

1) Buffa, E. S., "Models of production and Operations Management", John Wiley & Sons, Inc. New York, London 1963 p. 6

2) Berg, T. L. & Shuchman, A., "Product Strategy and Management", (edt.) Holt, Rinehart and Winston Inc, 1963 p. 178

3) Churchman,W., Ackoff, R., Arnoff,L.,"Introduction to Operations Research", New York, John Wiley & Sons, Inc. London 1957 p. 279

The use of science in making decisions which were previously made by judgement alone can generally be described as Operations Research. An important subclass of programming problems is linear programming. Linear programming is one of the most powerful tools of operations research. (1)
It is concerned with solving a very special type of programming problem. (2) It deals with situations where the relation among the variables are linear i. e., there is a proportionality or simple multiplicative relationship between the units of the resources and the number of units produced.

Verbally, such a problem could be formed as follows:

> "assume that prescribed amounts of inputs of materials or efforts are available. Given the amount of the output, perunit and the return of a unit output, how how to determine certain combinations of inputs among the large number of different possible ways, in order to produce output with maximum return, where a linear relationship between the output andinput is assumed". (3)

From a historical point of view, there is an agreement that the linear programming technique was first presented and introduced by George B. Dantzig, Marshal Wood, and their associated in 1947, to assist in overall planning of the multitude of activities of the U. S. air force. (4)

---

1) Frazer, J.R., "Applied Linear Programming", Prentice-Hall, Inc., Englewood Cliffs, N. J., 1968 pp. 1 - 3

2) Hadley,G., op. cit. p. 4

3) Saaty,T., op. cit. p. 167

4) Gass, S. I., "Linear Programming", McGraw Hill, New York, 1969, p. ix

The initial mathematical statement of the general problem of linear programming was made by Dantzig in 1947 along with the simplex method, a systematic procedure to solve the problem. (1)

The term linear programming (LP) was suggested to Dantzig by T. C. Koopmans in 1951 as an alternative to the earlier form, "programming in a linear structure". (2)
Dantzig's words for introducing the essence of this research in this area, "...that interrelations between activities of a large organization be viewed as linear programming type model and the optimizing programm determined by "minimizing" a linear objective function". (3)
Dantzig considers the introduction of LP in economics to be an anachronism. He thinks it would have arisen logically around 1758 when economists first began to describe economic systems in mathematical forms. Indeed, a crude example of LP can be found in the Tableau Economique of Quesnay. Also, we find that Walras proposed a sophisticated model in 1974 which had fixed technological coefficients as part of its structure. (4)
Prior to the World War II, already in 1939, the Soviet mathematician L. V. Kantorovich published his "Matyematicheskiye Metody Organizatsii Proizvodstava" (Mathematical Method of production Planning and Organization) in which he exposed the basic ideas of the science of programming in relation to problems of organization of production and transport. (5)

---

1) Dantzig, G. B., "Maximization of a Linear Function of Variables Subject to Linear Inequalities", in T. C. Koopmans (ed.), Activity Analysis of Production and Allocation, John Wiley & Sons, Inc., New York, 1951, pp. 339-347

2) Dantzig, G. B., "Linear Programming and Extentions", Princeton University Press, Princeton, N. J., 1963, p. 6

3) Gass, S. I. , op. cit. p. ix

4) Dantzig, G. B., Linear Programming..., op.cit. o.16

5) Lange, O. , "Optimal Decisions, Principles of Programming", Percamon Press Oxford, New York, Toronto, Sydney, Braunschweig, 1971, p. 6

"Kantorovich should be credited with being the first to recognize that certain important broad classes of production problems had well-defined mathematical structure which, were amenable to practical numerical evaluation and could be numerically solved". (1)

In 1941 the diet problem, an important example of linear programming, was formulated and approximately proposed by Jerome Cornfield. In 1945, George Stigler (2) treated the same problem which he calls 'experimental procedure'.(3)

Another important form of linear programming, transportation problem, was formulated by Hitchcock in 1941,(4) independently by Kantorovich in 1942, and by Koopmans in 1947. In 1947 Dantzig and Laderman solved the Stigler diet problem by use of the simplex method. They found that Stigler's solution was only one iteration away from the optimal solution. (5)
It soon became clear that linear programming had applications which went far beyond the military sphere. Charnes and Cooper pioneered the application of linear programming to industrial problems of planning and production. Koopmans, Samuelson, Georgeson, Roegen, and others began to apply LP to Leontief's input-output models of economic systems,while Kuhn, Tucker, and other mathematicians began to study important connections between LP and von Neumann's Theory of Games. (6)

In the next chapter, we will try to make a brief outline of the general LP problem and discuss the most important methods devoted to solve the linear programming problems.

---

1) Dantzig, G. B., "Linear Programming...", op. cit. p. 22

2) Stigler, G. J., "The Cost of Subsistance", Journal of Farm Economics, Vol. 27, 1945

3) Ackoff, R. L., op. cit. p. 112

4) Hitchcock, F.L.,"The Distribution of a Product from Several Sources to Numerious Localities", Journal of Math. and Phy., Vol. 20, 1941, p. 224-230.

5) Ibid

6) Garvin, W.W., "Introduction to Linear Programming", McGraw-Hill Book Company, Inc., New York, 1961, p. viii

## 2. Existing Methods of Linear Programming: brief outlines and cretiques.

### 2.1 THE GENERAL LINEAR PROGRAMMING PROBLEM

Linear programming is a systematic technique or process which is capable of optimizing a system as a whole, considering one objective function, rather than optimizing a part or parts of the whole. (1)

The "linear" in linear programming implies that only linear relations are involved. The "programming" in the term indicates that different variables are to be programmed - programmed in the sense of being scheduled or selected - to optimize a linear objective function. (2)

The linear programming problem, when translated into purely mathematical terms, requires a method for determining a solution to a system of simultaneous and linear inequalities which optimizes a linear function. (3) This problem may be stated symbolically as follows: Find a vector $x_1, x_2, \ldots, x_n$ which maximizes the linear form (objective function):

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \qquad \ldots(i)$$

subject to the constraints:

---

1) di Roccaferrera, "Introduction to Linear Programming Processes", South Western Publishing Company, Cincinnati, Ohio, 1967, p. 195

2) Pierre, D. A. ,"Optimization Theory with Application", John Wiley & Sons, Inc., 1969, p. 193

3) Dantzig, G. B., "Linear Programming and Extensions", Princeton University Press, Princeton, N. J., 1963, p. 6

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \quad \begin{matrix}\geq \\ < \end{matrix} \quad b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \quad \begin{matrix}\geq \\ < \end{matrix} \quad b_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \quad \begin{matrix}\geq \\ < \end{matrix} \quad b_m \quad \dots(ii)$$

and

$$x_1 \ , x_2 \ , \dots, x_n \geq 0$$

where the $a_{ij}$ and $b_i$ and $c_j$ are given constants, and only one of the signs $\leq \ \geq \ =$ holds for each constraint, but the signs can vary from one constraint to another. (1) Geometrically, the solutions to the constraints (ii) constitute a convex set§, and the optimum solution which minimizes or maximizes the objective function (i) is an extreme point of the set§§ (2) In a convex set defined by linear inequalities the optimal extreme point is the intersection of two lines in a 2-dimensional space. If n=3 the extreme points would be the intersection of three planes in an 3-dimensional space. In general, in an n-dimensional space, the extreme points of such a convex would be intersections of n hyperplanes, ahyperplane being a linear equation of the type,

$$a_{11}x_1 + a_{12}x_2 + a_{1n}x_n = b_1 \qquad (3)$$

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1) Hadley,G., op. cit. p. 4

2) Chung, An-min, "Linear Programming" Charles E. Merril Books Inc., Colombus, Ohio, 1963, p. 66

3) Ibid, p. 70

§ A convex set as a collection of points such that, if x and y are any two points in the collection, the straight line segment joining them is also in the collection.

§§ An extreme point of a convex set is one which does not lie on a line segment joining some two other points of the set. (Russel L. Ackoff op. cit. p. 125)

The intersection of these hyperplanes forms another hyperplane, one of lower dimension, and the values of x which lie on this hyperplane and which satisfy the non-negativity condition constitute a convex set. This convex is generally polyhedron and the vertices of the convex have at most m non-zero components (where m is the number of the constraints (ii) ). The vertices of the convex are of great value because the optimal solution is obtained at one of them. (1)

The linear programming problem can be, then, described as follows: " we are given a convex set which is defined by a set of linear constraints in n-dimensional Euclidean space. From all the points that belong to the convex set, we wish to determine a subset of points for which a linear objective function is optimized ".(2)

So constituted, the problem is completely solved from a theoretical point of view as long as the search for a maximum turns out to be a search over the vertices of the convex set. It should be noted, however, that when dealing with practical problems, the approach which consists of testing each vertex and picking up the one which maximizes the objective function must be out of the question.(3) There are two very good reasons why such an approach is not practical. First, because the actual number of the extreme points may be rather large. The intersection of k linear equations in n unknowns could form a convex set in $E^n$ with as many as

$$\binom{n}{k} = \left(\frac{n!}{(n-k)!\ k!}\right) \quad \text{extreme points.}$$

Thus for example, a system of 10 equations in 20 unknowns could form a convex set with

---

1) Pierre, D. A., op. cit. p. 200

2) Gass, S.I., op. cit. p. 37

3) Pierre, D. A., op. cit. p. 204

$$\binom{20}{10} = 184756$$

extreme points. (1) §

Such a procedure is also objectionable in that it does not indicate when a problem has an unbounded solution. If a problem has an unbounded solution, we could not detect this fact and would instead select the alternative with the largest value of the objective function as an optimal solution. In short, such an approach would lead to incorrect solutions if the problem has an unbounded solution. (2)

What it is obviously required, therefore, is an effective technique or algorithm with which to progress from one vertex to another so that the second vertex is guaranteed in advance to have an objective function of greater value.

In short the procedure we need should enable us to perform the following steps:

a) finding at least one vertex.
b) deciding whether it is the vertex that yields the optimal solution or not.
c) if it is not, finding another vertex that is better.
d) a guarantee that if steps b and c are repeated they will eventually lead to the optimal extreme point.
e) a device to show if the problem being solved is unbounded or even infeasible. (4)
f) a guarantee that the value of any extreme point will be better than the preceding one.

---

1) Cooper & Steinberg, "Introduction to Mathematics of Optimization", W. B. Saunders Company Philadelphia, London, Toronto, 1970, p. 77

2) Hadley,G. & Kemp,M.C., "Finite Mathematics in Business and Economics", North-Holland Publishing Company, 1972, p. 330

3) Pierre, D. A., op. cit. p. 204

§ Indeed, the actual number of extreme points is currently unknown. Current research (Grünbaum,B.,"Convex Polytopes", Interscience Publishers, New York, 1967) indicates that it may be somewhat less than $\binom{n}{k}$

4) Dean, B. V., Sasieni, M. W. and Gupta,S.K. , "Mathematics for Modern Management", John Wiley & Sons, Inc., New York and London, 1963, p. 348

By never reducing the value of the objective function, the need to investigate all the extreme point solutions is eliminated. Normally, only a very small fraction needs to be examined before an optimal solution is found. (1)

It has not been possible, thus far, to find an analytical procedure to solve such a problem. (2) "No method has been found which gives an optimal solution in a single step". (3)

Since the more efficient techniques for solving systems of simultaneous linear equations are iterative, it is not surprising that iterative techniques § have been found to be the most efficient means for solving linear programming problems. (4)

"The inability to find an analytical method led to the development of some numerical methods which are capable of finding an optimal solution for given values of $a_{ij}$, $b_i$, and $c_j$". (5)

The best-known and the most widely used procedure for solving linear programming problems is the Simplex Method, (SM) , developed by George Dantzig in 1947. (6)

"Other computational techniques and variations of the simplex method are continuously being developed". (7)
Aside from the simplex method there are some other methods which have been developed by

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1) Hadley,G.& Kemp, M.C., op. cit. p. 331

2) Garvin, W. W., op. cit. p. 26

3) Hadley,G., op. cit. p. 19

4) Ibid
5) Ibid
6) Ibid
7) Dantzig, G.,op. cit. p. 25

§ By iterative techniques it is meant that the solution is carried out through a number of steps, each of which brings us closer to the desired value (R. P. Dorfman, P. Samuelson, and R. M. Solow, "Linear Programming and Economic Analysis", McGraw-Hill Book Company, New York, 1958, p. 667

Beale (1), Kelley (2), Motzkin and Schoenberg (3), Motzkin, Thompson and Thrall (4), Tompkins (5), von Neumann (6) George W. Brown (7), R. Frisch (8), and Agnon (9).

However,as indicated by Hoffman, Mannos, Sokolowsky, these latter methods have not proved as effective in solving linear programming problems as the simplex method or some of its variants - namely because of the slowness in convergence and the very large number of the iterations require to achieve an optimal solution. (10)

---


1) Beale, E.M.L. , "An Alternative Method for Linear Programming", Proc. Cambridge Philos. Soc., 50, 1954, pp. 513 - 532

2) Kelley, J. E., Jr., " A Threshold Method for Linear Programming", Nav. Res. Log. Quart., 4, 1957, pp. 35 - 45

3) Motzkin, T. S., and Schoenberg, I. J., "Relaxation Method for Linear Inequalities", Canadian J. of Math., 6, 1954, pp. 393 - 404

4) Motzkin, T. S., Raiffa, H., Thompson, G. L. and Thrall, R. M., "The Double Descri tion Method", in Kuhn and Tucker, (eds.), Contri utions to the theory of Games, 11, Anals of Math. Studies, No. 28, Princeton U. Press, Princeton, 1953, pp. 51 - 73

5) Tompkins, C., "Projection Methods in Calculation",in Antosiewicz, H. A., (ed.) Proceedings of the Second Symposium in Linear Programming, Directorate of Management Analysis and National Bireau of Standards, Washington, D. C., 1955, pp. 425 - 447

6) von Neumann, J., "A Numerical Method to Determine Optimum Strategy", Nav. Res. Log. Quart., 1, 1954, pp. 109-115

7) Brown, George W., "Iterative Solution of Games by Fictitious Play", 1951, Ch. 24 in Koopmans, T. C., (ed.) Activity Analysis of Production and Allocation, Cowles Commission Monomgraph No. 13, Wiley, New York, 1951, pp. 374 - 376.

8) Frisch,R., "The Multiplex Method for Linear Programming", Memorandum, Sosialökon. Inst. University of Oslo, Norway, 1958

9) Agnon, S., "The Relaxation Method for Linear Programming Inequalities", Canadian J. of Math,m 6, 1954, pp. 382 - 392

10) Hoffman, A., Mannos, M., Sokolowsky, D., and Wiegmann, N., "Computational Experience in Solving Linear Programming", J. of the Soc. for Industrial and Applied Math., 1, 1953, pp. 17 - 34

Aside from the exact methods, there are some other methods which give an approximate solution to linear programming problems. (1)

## 2.2. Graphic Method *

The graphic method is certainly practical when there are two variables. Regardless of how many constraints must be considered, a problem with two variables can always be depicted in a plane or in a two dimensional space.(2) Hence the graphical approach has only limited application to the industrial problems owing to the inherent restraint placed on the number of variables that itis capable of handling. The graphic method can certainly be extended to three-dimensional problems, but the geometry in such a case becomes so complex that considerable accuracy is apt to be sacrificed. "It is obviously impossible to solve problems consisting more than three variables by purely graphical means."(3). Though limited in applicability, because "Industriebetriebe mit nur zwei Produktarten findet man in der betrieblichen Praxis zwar nur selten", (4) graphical approach is very useful in providing insight into some of the abstract theoretical concepts. (5) It shows how the restrictions of the problem limit the possible solution and how the objective function determines the optimal solution. (6) It is highly suggestive, as well, of the two important properties of linear programming problems. One property is that the solution to the set of constraints of a linear programming problem forms a convex set, and the other is that the optimal solution which maximizes or minimizes the objective function is an extreme point of the set. (7)

If there are for example, two products which must be manufactured on three machines, we can determine graphically the most efficient utilization of available machine capacities.

---

*(The foot notes for this section appear on p. 14).

Consider the following problem:

$$\text{Max} \quad z = 4x_1 + 8x_2$$

subject to:

$$\text{I.} \quad 4x_1 + 2x_2 \leq 20$$

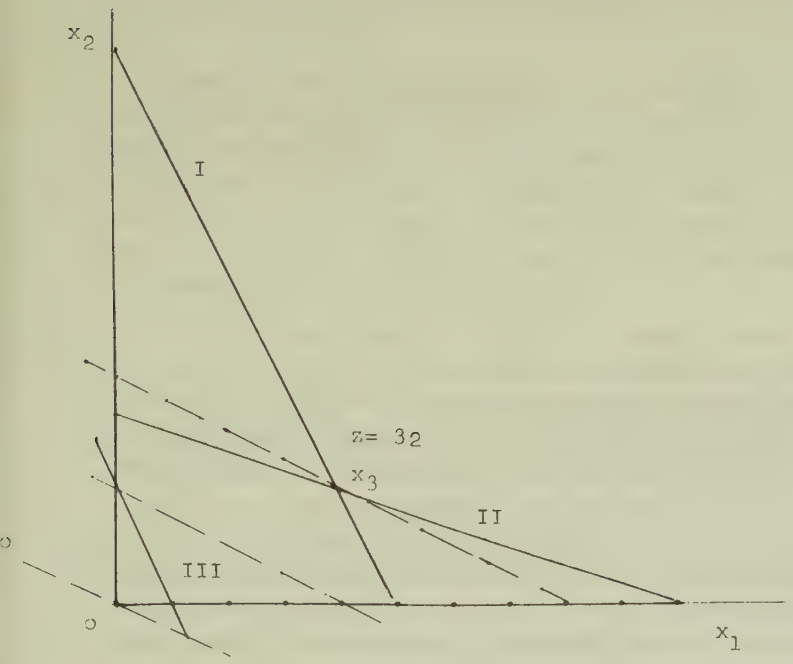
$$\text{II.} \quad x_1 + 3x_2 \leq 10$$

$$\text{III.} \quad 2x_1 + x_2 \geq 2$$

and

$$x_1, x_2 = 0$$

The problem could be presented graphically as follows:



where the dashed lines indicate the objective function with different values. It is easy to depict that the optimal solution lies on the point $x_3$ where the value of the objective function is 32 and the values of $x_1$ and $x_2$ are 4 and 2 respectively.

---

1) Tscheuschner, H. J., "Zwei Näherungsmethoden ökonomischer Natur zur Lösung von Aufgaben der Linearen Optimierung", in Wissenschaftliche Zeitschrift der Hochschule für Architektur und Bauwesen, Weimar, 14. Jg. Weimar, 1967, H. 3 pp. 302 - 304

2) di Roccaferrera, op. cit. p. 209

3) Thomas H. Naylor and Eugene T. Byrne, "Linear Programming, Methods and Cases", Wadsworth Publishing Compnay, Inc., Belmont, California, 1963, pp. 43 - 44

4) Wolfgang Kilger, "Optimale Produktions- und Absatzplanung", Westdeutscher Verlag, Opladen, 1973, p. 100

5) Chung An-min, op. cit. p. 61

6) Buffa, E.S., Modern Production Management, New York, 1965, p. 324

7) Chung An-min, op. cit. p. 66

## 2.3 SIMPLEX METHOD

Considering the following LP problem in its condensed form:

$$\max. \quad z = \sum_{j=1}^{n} c_j x_j \qquad \text{(i)}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \lesseqgtr b_i \qquad i = 1,2,\ldots,m \qquad \text{(ii)}$$

and

$$x_j \geq 0 \qquad j = 1,2,\ldots,n \qquad \text{(iii)}$$

the simplex method would be defined as an algorithm for determining values for a set of n nonnegative variables $x_j$ which minimizes or maximizes the linear form (i) subject to (ii) i. e., a set of m linear constraints. (1) The Simplex Method is considered to be "one of the most celebrated computational procedures in recent mathematical history". (2)

T. Saaty describes the simplex process geometrically as "one of several iterative procedures used in solving a linear programming problem. Each iteration of this process consists of translating the hyperplane corresponding to the objective function parallel to itself, each time evaluating its distance from the origin at the intersected vertex of the convex set defined by the inequalities (ii) i.e., the constraints. The process is repeated in such a way that the translated hyperplane in each step yields improved results leading to the optimum distance. The simplex process also involves a useful criterion which eliminates some

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1) Garvin W. W., op. cit. p. 26

2) Gale, D., "The Theory of Linear Economic Models", McGraw-Hill Book Company, New York, Toronto, London, 1960, p. 105

of the vertices of the convex set as possible trial points". (1)

A similar definition was given by di Roccaferrera, "The simplex method is an iterative process which approaches, step by step, an optimum solution in such a way that an objective function of maximization or minimization is fully reached. Each iteration in this process consists of shortening the distance (mathematically and graphically) from the objective function to the intercepted vertex of a convex set determined by the inequalities which describe the problem". (2)

The term "simplex" has nothing to do with the method as it is now used; it had its origins in a special problem that was studied in the early development of the method. (3)

In general mathematical terms, a k-dimensional simplex (sometimes called a closed simplex) is a set which consists of the following a) $k + 1$ elementary independent points $x_1 , x_2 , \ldots, x_{k+1}$ of an Euclidean space of dimension greater than K; and (b) all points of the type

$$x = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1,k+1}\, x_{k+1}$$

where $\sum_{i=1}^{k+1} a_{1i} = 1$

and where all $a_{1i}$'s are constrained by $\geq 0$ (4)

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1) Saaty, T. L., op. cit. p. 177

2) di Roccaferrera, op. cit. p. 423

3) Hadley, G., op. cit. p. 19

4) Pierre, D. A., op. cit. p. 205

To see the idea behind the method from a computational point of view, consider the relations (ii)

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \quad i = 1,2,\ldots,m \qquad n > m$$

They consist of m inequalities in n unknowns. Thus, if we arbitrarily give any n-m of the unknowns the value of zero, there will remain an ordinary set of m simultaneous equations in m unknowns. Such a set - under certain conditions - has a unique solution. Keeping in mind that selecting an arbitrary set of n-m from the original could be performed in a finite number of trials, it would be fair to say that such a process can, in finite steps, lead to the required solution. (1)

Interest is focused on those solutions which are obtained by setting n-m of the x's equal to zero. Such solutions are called basic solutions. If the remaining m of the x's of a basic solution are forced to be nonzero the solution is called a nondegenerated basic solution, but if one or more of the remaining x's can assume the zero value, the solution is called a degenerated basic solution. Furthermore, if a basic solution contains nonnegative x's, it is called a basic feasible solution - feasible in the sense that it satisfies the constraints. (2)

The strategy of the simplex process may be briefly described as follows: since each basic feasible solution is associated with a basis, the process starts with a basis that yields a basic feasible solution. It then moves on to a new basis associated with a better basic feasible solution by replacing one of the current basis vectors with a nonbasis vector.

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1) Dorfman, S. & others op. cit. p. 69

2) Pierre, D. A., op. cit. p. 200

This nonbasis vector is selected because of its potential contribution to the improvement of the objective function which can be defined in advance. On the other hand, the current basis vector which must leave the basis is selected according to a rule which insures the continuation of the feasibility of the new solution. (1)

If one thinks of this process geometrically, what we are doing is the following: Each basic feasible solution corresponds to an extreme point of the convex set of the feasible solutions. By replacing one vector at a time, we move from one vertex to an adjacent vertex of the convex set. (2)

It might be obvious that G. B. Dantzig has intelligently used the so-called Gauss-Jordan elimination method for solving a system of n unknowns in n equations. Instead of haphazard pivoting, he has combined this process with the objective function. Therefore, the selecting of the pivot elements follows a certain rule so that the objective function may be improved and the feasibility of the solution may be continued.

Thus, a step of the simplex method is a modified Gauss-Jordan step with the following rules for choice of the pivot:

1 - The pivot column is selected so as to have the largest (negative) element in the last row (the objective function row) (Steepest Ascent)

2 - The pivot row is selected so as to have the smallest among the positive values which are defined according to a certain rule. (3)

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1) Chung An-Min, op. cit. pp. 89 - 90

2) Cooper and Steinberg, "Introduction to Methods of Optimization", W. B. Saunders Company, Philadelphia, London, Toronto, 1970

3) Stiefel, E., "Note on Jordan Elimination, "Linear Programming and Tchebycheff Approximation", Numerische Mathematik, Vol., 2, 1960, pp. 1 - 17

"The simplex algorithm is a powerful analytical machine whose impact on decision making in business, industry, and government makes it one of the major achievements of applied mathematics in this century. ... However it became clear very soon after LP began to be applied to practical problems, that the simplex method is not a perfect machine." (1) First, the required computational procedures prove to be very time-consuming even when considered on the time scale of electronic digital computers. In the simplex we have to compute many numbers which either are never needed at all in the process or are needed only in an indirect way. (2)

Furthermore, it does not always make the best move in each iteration. At each iteration, one nonbasic variable is elevated from zero to some positive value. At another iteration this very elevated nonbasic may be removed from the basic solution to become once more a nonbasic variable. (3)

Dantzig acknowledges this fact; he said, "while the SM appears a natural one to try in the n-dimensional space of the variables, it might be expected, a priori, to be in efficient as there could be considerable wandering on the outside edges of the convex of solutions". (4)

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1) McMillan, C., Jr., "Math. prog.", John Willey & Sons, Inc., New York, 1970, pp. 30-31

2) Garvin, W.W., op.cit. p. 191

3) McMillan, C. Jr., op. cit., pp. 30-31

4) Dantzig, G., "Linear Programming...", op. cit. p. 160

Another possible inefficiency in the simplex process is the need for an initial basic feasible solution to start with. If there is no such solution in the original problem, a certain kind of slack and artificial variables must be used. Such a fact leads, indeed, to the necessity of solving another linear problem.

Another point is that, as the simplex method progresses from one basis to another, it often happens, particularly in large problems, that the inevitable small round-off errors introduced at each stage may accumulate to a point where the accumulated round-off errors become so "significant as to yield false results. For example, a non-optimum solution may appear to be optimum (due to the errors in the criterion function $(z_j - c_j)$ while, on the other hand, an optimum solution may show up as non-optimum" and hence additional iteration may be created. (1) Unfortunately, "there does not seem to be a sound way of estimating a priori the amount of round off error in the solution". (2)

Another disadvantage of the simplex method is that the original coefficient matrix contains a considerable number of zeros, especially if the matrix is of large dimension. Dantzig considers the percentage of the zeros to be more than 90 per cent. (3) This property has for a long time been used

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1) Ackoff, R. L., op. cit. pp. 130 - 131

2) Wolfe, P., "Error in the solution of Linear Programming Problems", in Luis B. Rall (ed.) Errors in Digital Computation, Vol. 2, Joh. Wiley & Sons, Inc. 1965, pp. 271 - 285

3) Dantzig, G., "Linear Programming...", op. cit. p. 210

to reduce storage requirements for the data, in other words, to save memory space in computers by keeping only the non-zero data. (1)

Unfortunately, as the process progresses, the transformation procedure changes some zero elements to non-zero and, hence, creates a substantial amount of computation and necessitates a substantial amount of the storage capacity. (2)

These disadvantages lead to the development of the so-called Revised Simplex Method as a scheme for ordering the computations required so that unnecessary calculation are avoided.

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1) di Roccaferrera, op. cit. p. 563

2) Simmonard, M., "Linear Programming", Prentice-Hall, Englewood Cliffs, N. J., 1966, p. 86

## 2.4 THE REVISED SIMPLEX METHOD

Extensive application of the standard simplex method has called for improvements in several directions: (1)

a) Some device has to be made in order to minimize the computational efforts.

b) Some device has to be made in order to avoid the problem of the accumulated rounding-off-errors

c) Some device has to be made to exploit the existence of the high percentage of zeros in the original matrix.

To resolve such problems, Charnes and Lemke (2) developed a modified form for the original simplex procedure. The new modification came to be known as The Revised Simplex Method or RSM.

The RSM, like the original process, is a chain of steps in which the data entered in the computation are obtained from the previous set of data. It differs from the [illegible] method in that it uses the same recursion relations to transform only the inverse of the basis for each iteration. (3) There is a kind of selection made in order to carry on only those elements which are needed in the next iteration.

It has been found that the essential element which enables us to progress from a certain basic solution to another is the explicit knowledge of the representation of the nonbasic vectors in terms of the basis vectors. (4)

1) Ackoff, R. L., op. cit. p. 181
2) Charnes, A., and Lemke, C. E., "A Modified SM for Control of Round-off Error in LP"(mimeographed), Carnegie Institute of Technology, May 7, 1952
Lemke, C. E., and Charnes, A., "Control of Round-off Error in LP", 35 - 38, In Extremal Problems in L Ineq. Technical Report, No. 36, Carnegie Institute of Technology, Pittsburgh, 1953
3) di Roccaferrera, op. cit. p. 561
4) Gass, S. I.I, op. cit. p. 106
Judin, D. B., Golstein, E. G.: Lineare Optimierung I, Berlin 1968, p. 342.

The revised simplex method could be described as follows:

let the inverse of a current basis be $B^{-1}$, and the current solution be $X = B^{-1} b$

To move towards another extreme point we may:

Step 1 : calculate the current relative cost coefficients

$$\beta_j = C_B B^{-1} a_j - C_j \qquad j = 1,2,\ldots,n$$

where

$$C_B = (c_1 , c_2,\ldots,c_m)$$

is a row vector.

If $\beta_j \geq 0$ , for all j, j=1,2,...,n. then the current solution is optimal.

Step 2: determine which vector $a_e$ is to enter the basis by selecting the most negative cost coefficients $\beta_j$ , let it be $\beta_e$, which means that the vector $a_e$ is to enter the basis. Then calculate $y_e = B^{-1} . a_e$ which is the vector $a_e$ expressed in terms of the current basis.

Step 3: calculate the ratio $P_i^o / y_e$ to determine which vector is to leave the basis

Step 4: perform the transformation procedure to define the inverse of the current solution. Return to step 1. (1)

In order to perceive the idea behind the revised form, let us assume that a particular variable $X_k$ never enters any of the bases that are generated during the course

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1) D. Luenberger, "Introduction to Linear Programming", Addison-Wesley Publishing Company, Reading, Mass., 1965, p. 55

of the iterations. The calculation of the coefficient $y_{ik}$, i. e., the value of $X_k$ in terms of the succesive basic solutions, could be considered as a waste of time, because the coefficients $y_{ik}$ are never needed directly in any way but are required only indirectly in the determination of $(Z_j - C_j)$ for each iteration. On the other hand, suppose that $X_k$ enters the basis during certain iteration say the 99 th iteration. Then the transformation of the coefficients for the last 98 iterations could be considered as a waste of time as well since we need only the $y_{ik}$ in the 99 th iteration. The trouble is that the computational process looks like a chain which cannot be broken at any time (1)

In short, the RSM could be considered as a practical answer for the next two questions:

a) Is it possible to determine $(Z_j - C_j)$ in some manner which does not involve the transformation of the coefficients of the nonbasic variables at every iteration?

b) Is it possible to determine the value of any nonbasic variable in terms of the basic variables only whenever they are needed?

In order to give more insight into the philosophy of the RSM, let us recall the content of the simplex tableau T at any iteration (2) where the initial problem is as follows:

| | |
|---|---|
| Max. | $CX$ |
| subject to | $AX \le P^0$ |
| and | $X \ge 0$ |

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1) Garvin, W. W. , op. cit. p. 191

2) Chung, An-min, op. cit. p. 148
Krekó, B., Lehrbuch der Linearen Optimierung, [illegible] Deutscher Verlag der Wissenschaft, Berlin, [illegible]
Judin, Golstein, op. cit. 342

$$T = \begin{bmatrix} Y & \overline{X} \\ Z - C & F(X) \end{bmatrix} \quad \ldots\ldots\ldots\ldots (1)$$

which is the same as (1)

$$T = \begin{bmatrix} B^{-1} A & B^{-1} P^{o} \\ CB^{-1} A - C_j & CB^{-1} p^{o} \end{bmatrix} \quad \ldots\ldots\ldots\ldots (2)$$

where A is the original matrix, $B^{-1}$ is the inverse of the current basis, $X = B^{-1} p^{o}$ is the value of the current solution, and $CB^{-1}$ is the relative cost coefficient.

The last expression is nothing more than the product of the following two matrices:

$$\begin{bmatrix} B^{-1} & 0 \\ CB^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} A & P^{o} \\ -C & 0 \end{bmatrix} \quad \S \quad \ldots\ldots\ldots\ldots (3)$$

$$= \begin{bmatrix} B & 0 \\ -C & I \end{bmatrix}^{-1} \cdot \begin{bmatrix} A & P^{o} \\ -C & 0 \end{bmatrix} \quad \ldots\ldots\ldots\ldots (4)$$

$$T = B^{-1} A \quad \ldots\ldots\ldots\ldots (5)$$

where

$$B^{-1} = \begin{bmatrix} B & 0 \\ -C & I \end{bmatrix}^{-1}$$

and $$A = \begin{bmatrix} A & P^{o} \\ -C & 0 \end{bmatrix}$$

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§) For the proof consult Chung, An min, op. cit. , pp. 146-148
Krekó, B.: op cit ,p. 392

Thus we see that each simplex tableau is the product of two matrices. The first matrix ($B^{-1}$) is essentially the inverse of the basis (B) and the second (A) is nothing more than a complete tabulation of the original coefficients in the objective function and the system of the constraints. So constituted, it is obvious that A is not affected by change of basis. Herein lies the basic difference between the regular and the revised simplex methods. This means that in the normal form the entire simplex tableau is transformed as we move from one basic feasible solution to another, where as in the modified form only the transformation of $B^{-1}$ is required. Since the tableau is usually larger in dimension than the inverse $B^{-1}$, it is apparent that the RSM offers important advantages over the regular simplex. Moreover, the smaller matrix stored in the internal memory of the computer now enables us to handle larger problems.

There is a commonly held misconception that the revised simplex method requires less computational effort, (Simmonard, Chung, Gass)*. This is not precisely true in the manner in which it is generally implied. There are advantages of precision and storage which make the revised simplex method very attractive, but the decrease, if any, in the amount of computation is not a major factor.(2) Applying the revised form, control of round-off error is located to control of errors of the inverse of the basis at each iteration. (3) Dantzig explains how the revised form is advantagious from the storage capacity point of view; he says, "Less machine memory is needed using multiplier method for recording because the original coefficients are often given in fixed decimal of three to five places. This is considerably less than that required for $a'_{ij}$ when the Standard Simplex method

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1) Ibid
2) Cooper & Steinberg, op. cit. p. 239
3) Ackoff, R. L. , op. cit. p. 131

* see Garvin, W.W., op. cit., p. 203, Gass, I.S., op. cit. pp. 106-120, Simmonard, M., op.cit. p. 88

is used, for this avoids round off error difficulties in the passage from one iteration to another.". (1)

(1) Dantzig, G., "Linear Prog. ..." op cit p. 216

## 2.4.2 The Product Form of the Inverse

The real practical problems always assume rather high dimensions, a problem with 500 rows and 1000 columns is not a large problem anymore. For such a problem 500.000 storage positions are required. This does not include working storage or programm storage. High-speed storage of this size is scarcely available; thus auxiliary storage such as magnetic tapes, disk, or drum must be used. However, since using the auxiliary storage costs unneglectable long operational time, it is worth considerable effort to reduce the amount of data which must be transmitted between high-speed and auxiliary storage. Each tableau must be at least read, transformed, and written at each iteration. (1)

Dantzig stated that, A. Arden, in the early days of LP found that it might be convenient from a computational point of view, to represent the inverse of the basis as a product of elementary matrices. (2)

By elementary matrix is meant one formed from the identity matrix by replacing one column. Thus, an elementary matrix can be compactly recorded by the subscript of the altered column and the values of the elements in it. (3)

To understand the idea behind the new alternative, let us assume that at the $K^{th}$ iteration the inverse $B^{-1}$ is transformed by a pivot operation where the pivot column is

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1) Orchard-Hays,W. , "Advanced LP Computing Techniques", McGraw-Hill Book Company, New York, San Francisco, Toronto, London, Sydney, 1968, p. 63
2) Dantzig,G. , "Linear Programming...", op.cit. p. 200
3) Dantzig,G. , "Product Form of Inverse", Tables and other aids to Computation, Viii/46 April, 1954, pp. 64 - 65

$$y'_j = (y_{1j}, y_{2j}, \cdots, y_{mj})$$

and the pivot element is $y_{gj}$.

The result of the pivot operation is the new matrix

$$E\, B^{-1}$$

where E is the elementary matrix E.

$$E = \begin{bmatrix} 1 & & & v_1 & \\ & 1 & & v_2 & \\ & & \ddots & \vdots & \\ & & & v_g & \\ & & & \vdots & 1 \\ & & & v_m & & 1 \end{bmatrix}$$

and where $v_i = - y_{ij} / y_{qj} \quad i \neq q$

$v_q = 1 / y_{qj}$

If we start the simplex procedure with a basis that is the identity matrix, then after the $n^{th}$ pivot the inverse of the basis matrix is

$$B^{-1} = E_n E_{n-1} \ldots E_2 E_1$$

where $E_i$ is the elementary matrix corresponding the $i^{th}$ pivot operation. (1)

When problems are solved on digital computers, the inverse is usually stored on tape,(2) since it might be sometimes impossible to store the whole inverse $B^{-1}$ in the fast memory of the computer.(3)

Then, there are advantages to be gained if one never specifically computes the inverse at every iteration, but instead represent it in its product form, i. e., as the product of the E -matrices. After the $K^{th}$ iteration the inverse will be represented as the product of K, E-matrices.

When the original matrix $B^{-1}$ is too large to be stored internally, the pivot matrices are stored in some auxiliary memory, and when needed, they are called one after the other and the multiplications are performed in sequence. (4)

It is not necessary to store each elementary matrix in full form, since most of its elements are zero- it is only required to store the new column's and its index . (5)

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1) Luenberger,D.G. , op. cit. p. 59
2) Hadley,G., op. cit. p. 217
3) Sakarovitch,M., "Notes on Linear Programming", van Nostrand Reinhold Company, New York, London, 1971, p. 103
4) Sakarovitch,M., op. cit. p. 104
5) Luenberger,D., op. cit. p. 59

In other words, the essential part of the elementary matrix inverses are recorded as a string of data on a sequential storage device such as a magnetic tape. This can be depicted as follows:

$$r_1 \; ; y_{11}, \ldots, y_{1m} ; \; r_2 \; , y_{21}, \ldots, y_{2m}, \ldots, r_k, y_{k1}, \ldots, y_{km} \; .$$

where semicolons seperates iterations,

and the index of the pivot position heads each sublist from an iteration. (1)

Thus,the steps required to be performed are as follows:

Step 1: Calculate the current basic solution by the recursive formula

$$B^{-1} = ( E_k(E_{k-1} \ldots (E_1 P^0)))$$

Step 2: Calculate the current relative cost coefficients

$$\beta_j = C_B \; B^{-1} \; a_j - c_j$$

by using the form

$$\beta_j = \alpha \; a_j - c_j \quad \text{where} \quad \alpha = C_B \; B^{-1}$$

The row vector $\alpha$ is first calculated by the recursive formula

$$\alpha = (((\; C_B \; E_k) \; E_{k-1}) \ldots E_1)$$

If $\beta \geq 0$ , stop; the current solution is optimal

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1) Orchard-Hays,W. , "Advanced Linear Programming Computing Techniques", McGraw-Hill Book Company, New York, London, 1968, p. 70

Step 3: Select the most negative relative cost coefficient $\bar{c}_j$ and calculate the value of the incoming variable (vector) in terms of the current basis.

$$y_j = (E_k\ (E_{k-1} \cdots (E_1\ a_j)))$$

Step 4: Calculate the ratios $p_i^o / y_{ij}$ to determine which vector is to leave the basis - then determine which element is the next pivot and hence define $E_{k+1}$. Return to step 1 (1)

Reinversion:

When using the product from the inverse, another column is generated on every change of basis. Hence, the product form of inverse can be become rather long. "This is clearly undesirable as it will ultimately exceed the explicit inverse in size"(2) .

It follows, then that it would be adventageous as the number of iterations approaches or perhaps exceeds the number of the constraints, to reinvert the original elements of the basis and to write in memory only the non-unit columns of the matrix product obtained. (3) Graves and Wolfe stated that "Markowitz"suggests an optimum reinversion policy as to maintain the average time per iteration at a minimum (4)

It might be clear, then that the major advantage of generating the inverse for each basis,by means of elementary matrices, is that only a minimum amount of information,

1) Luenberger,D., op.cit. p. 59
2) Orchard-Hays,W., op.cit. p. 73
3) Simmonard,M., op. cit. p. 87
4) Graves,R.L., and P. Wolfe, "Recent Advances in Mathematical Programming", McGraw-Hill Book Company, New York, 1963 , p. 213

namely the elements of the elementary matrix and its subscript, need to be recorded. (1) This is extremely important when we must use a computer with a limited high speed storage. Moreover, the product form of the inverse requires fewer computations than the other SM variants. (2)

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1) Gass,S. , op. cit. p. 125

2) Luenberger,D. , op.cit. p. 58

# 3. KAIRO METHOD

## 3.1 Introduction

Techniques of efficient computation are all interested in the solution of one major problem, namely, "how to obtain a chain of non-negative vertex solutions with the least effort such that, each time a new solution is obtained, there is an improvement in the value of the objective function". (1) The computational effort required for solving practical LP proved always to be formidable, even when digital computers, are used. Another challenge to the solution of practical LP problems is the growing size of the problems to be considered. It becomes worth considerable effort to reduce the amount of the required computation. A third difficulty faced by the programmer is the accumulation of the round-off error specially in large problems.

We have seen, however, how the RSM and the product form of the inverse presented a successful solution for the previous challenges.

The basic achievement of the revised simplex method is that we are able to determine the value of a certain activity not in the current basis in terms of the basic vertex only whenever it is needed. (2) Furthermore, it benefits from the existence of the high percentage of zero coefficients.

On the other hand, the major advantage of applying the product form of the inverse is that only a minimum amount of information needs to be recorded. (3)

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1) Ackoff, R. L., op. cit. p. 162

2) Dorfman & others, op. cit. p. 84

3) Garvin, W., op. cit. pp. 191-203

While the SM or any of its variants deal with the problem as it is, there is another approach which aims to minimize the size of the problem being considered by means of eliminating the dominated activities i.e., the activities which use the resources ineffieciently - and the dominated rows, i.e., the constraints which never actually limit the production. (1)

In the last few years, several methods have been developed for locating the dominated rows and columns. (2) Such approaches, however, increase the tiresome computational burden of solving a LP problem. Furthermore, it is not always easy to recognize the redundancy. (3)

It is interesting to note that the RSM could be considered as a partial solution for the problem of dominated columns, since it performs a certain kind of selection for the columns to be transformed, and hence, avoids the necessity of transforming and recording the coefficients of the nonbasic variables during the course of the iterations.

## 3.2 The Revised Simplex : criticized

A deeper examination of the inverse generated at any iteration, when using the RSM, would reveal the following facts:

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1) E. Heady & W. Candler "Linear Programming Methods", The Lower State College Press, Ames. Iowa, USA, (1959), pp. 151-155

2) Ibid p. 152

3) Levin,R.S.& Lamone , R.O., "Linear Programming for Management Decisions,", Illinois, Irwin Inc., 1969 p. 196

a) The successive relative cost coefficients, or the simplex multipliers §, as they are usually named, are only determined through:

i) the outgoing row vector (s).

ii) the successive new pivot(s), and

iii) the current simplex multipliers.

To make this point clear, let the following tableau be the inverse of the initial basis at the first iteration

$$\begin{array}{cc} x_{n+1} \cdots\cdots x_{n+m} & y_j \\ \left(\begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \\ & & \\ 0 \ldots 0 & \ldots & 0 \end{array}\right) & \begin{array}{c} y_{1j} \\ \vdots \\ y_{ij} \\ \vdots \\ y_{mj} \\ \\ \beta_j \end{array} \end{array}$$

where the additional vector

$$y_j = y_{1j}, y_{2j}, \cdots, y_{mj}$$

indicates the value of the newcoming variable $x_j$ in terms of the current basis and $\beta_j$ implies the relative cost coe-

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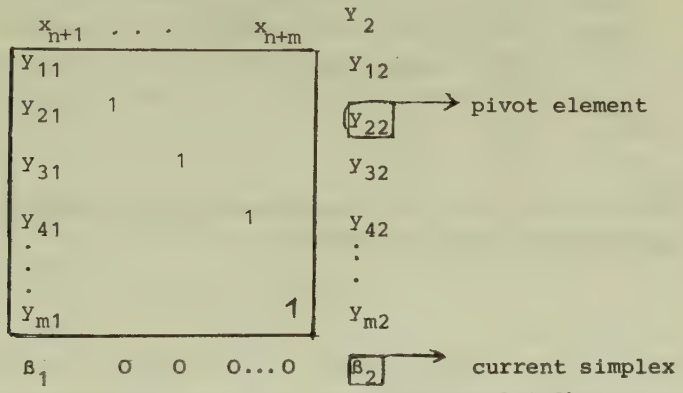
§ The reason for this term is that the simplex multipliers can be interpreted as prices; multiplying these prices by the input-output coefficients of an activity and summing evaluates or "prices out" the activity in terms of the substitute processes, the basic activities. The sum when compared with the direct cost $C_j$, tells us whether or not it pays to consider introducing a nonbasic activity j into the basic set". (Dantzig, G.B., "Linear Programming...", op.cit. p. 216)

ficient of the variable $x_j$. *

Let us assume that the first three real variables $x_1, x_2$ and $x_3$ replace successively the first three slack variables $x_{n+1}$, $x_{n+2}$ and $x_{n+3}$. The tableau at the second iteration would be as follows:



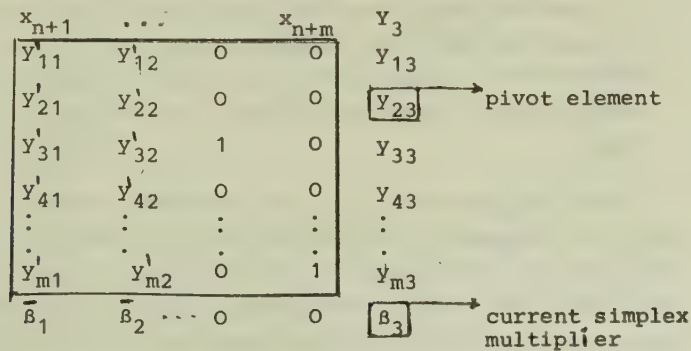

Updating the simplex multipliers $\bar{\beta}_1$ and $\bar{\beta}_2$

$$\bar{\beta}_1 = \beta_1 - y_{21} \cdot \beta_2 / y_{22} \qquad \text{(column 1)}$$

$$\bar{\beta}_2 = 0 - \beta_2 / y_{22} \qquad \text{(column 2)}$$

and transforming the elements $\sum_{1}^{m} Y_{ir}$ $r=1,2.$

the tableau at the third iteration would be as follows:




* notice that at the first iteration $Y_j = a_j$ and $\beta_j = c_j$ since the 'initial' inverse of the basis is still an identity matrix.

Updating the simplex multiplyers $\bar{\beta}_1$, $\bar{\beta}_2$ and $\bar{\beta}_3$

$$\bar{\beta}_1 = \beta_1 - y_{31} \cdot \beta_3 / y_{33}$$

$$\bar{\beta}_2 = \beta_2 - y_{32} \cdot \beta_3 / y_{33}$$

$$\bar{\beta}_3 = 0 - \beta_3 / y_{33}$$

and $$\bar{\beta}_d = 0 \qquad d = 4,5,\ldots,m$$

We may notice that the deriviation of $\bar{\beta}_1$, $\bar{\beta}_2$ and $\bar{\beta}_3$ is not affected with the elements $y_{dr}$ $d = 4,5,\ldots,m$ $r = 1,2$

Thus, if the optimal solution is obtained at the Kth iteration, it could be said that the transformation of the elements $y_{dr}$ $d = K+1, K+2,\ldots,m$ would be superfluous because they don't affect the successive values of the simplex multipliers.

b) In the case of the existence of dominated rows, the transformation of the successive coefficients of those dominated rows would be a waste of time since they never influence the solution.

c) If a certain row leaves the solution at the 99th iteration, it could be said that computing the coefficients of this row for the preceding 98th iterations is a waste of time, since we need these coefficients only at the 99th iteration.

But the problem is that the required coefficients at the "Kth" iteration are derived from the coefficients of all the previous iterations. So, we have to consider the whole set of the constraints because we do not know ahead of time which constraints are really needed and which are superflous.

If there is a way which enables us to determine the coefficients of any row and the value of the newcoming variable in terms of the current basis which does not involve the transformation of the coefficients ($y_{dr}$) for all the preceding iterations, or in other words, if we were able to perform a selective procedure for the rows, so that only the rows which really affect the solution need to be transformed; the computational procedure, certainly, would be smaller. This is, however, what KAIRO METHOD (KM) will try to do.

## 3.3 KAIRO METHOD, its basic argument:

Consider the following LP problem: §

$$\text{Max. } f(x) = C'X$$

subject to :

$$AX \leq p^{o}$$

and $X \geq 0$

where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \cdot & & \cdot \\ a_{m1} & & a_{mn} \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

(§) For the sake of simplicity, we we will assume that all the constraits are of the form $\leq$ .

$$P^o = \begin{pmatrix} p_1^o \\ p_2^o \\ \cdot \\ \cdot \\ \cdot \\ p_m^o \end{pmatrix} \qquad \text{and } C = (c_1\ c_2\ \ldots c_n)$$

Next we augument the system by the variables

$$x_{n+1} \quad x_{n+2} \quad \cdots x_{n+m}$$

which are constrained to be nonnegative. The variables $x_{n+1} \quad x_{n+2} \quad \cdots \quad x_{n+m}$ are called slack, artificial, or disposal variables because they do not belong to the original system. They are introduced because they constitute an immediate basic feasible solution.

The information of the problem could be presented in a tableau known as tableau of detached coefficients, where each row corresponds to an equation. The variables appear as column headings and the vertical bars that separate the columns perform the rules of plus and minus.

| $x_1$ | $x_2 \cdots x_j$ | $\cdots$ | $x_n$ | $x_{n+1}$ | $\cdots$ | $x_{n+m}$ | |
|---|---|---|---|---|---|---|---|
| $a_{11}$ | $a_{12} \cdots a_{1j}$ | $\cdots$ | $a_{1n}$ | $a_{1,n+1}$ | $\cdots$ | $a_{n+m}$ | $p_1^o$ |
| $a_{21}$ | $a_{22} \cdots a_{2j}$ | $\cdots$ | $a_{2n}$ | $a_{2,n+1}$ | $\cdots$ | $a_{n+m}$ | $p_2^o$ |
| $\cdots$ | $\cdots$ | | $\cdots$ | | | | $\vdots$ |
| $a_{m1}$ | $a_{m2} \cdots a_{mj}$ | $\cdots$ | $a_{mn}$ | $a_{m,n+1}$ | $\cdots$ | $a_{n+m}$ | $p_m^o$ |
| $c_1$ | $c_2 \quad c_j$ | $\cdots$ | $c_n$ | $c_{n+1}$ | $\cdots$ | $c_{n+m}$ | |

let us put the tableau into the following form: *

| | $x_r$ | $x_s$ | $x_f$ | $x_h$ | |
|---|---|---|---|---|---|
| $x_f$ | $A_{rr}$ | $A_{rs}$ | $A_{rf}$ | $A_{rh}$ | $p_r^o$ |
| $x_h$ | $A_{dr}$ | $A_{ds}$ | $A_{df}$ | $A_{dh}$ | $p_d^o$ |
| | $c_r$ | $c_s$ | $c_f$ | $c_h$ | |

where:

$A_{rr} = a_{rr}$ $\quad r = 1,2,\ldots,k.$ (square nonsingular matrix)

$A_{dr} = a_{dr}$ $\quad r = 1,2,\ldots,k$
$d = (k+1),(k+2),\ldots,m.$

$A_{rs} = a_{rs}$ $\quad r = 1,2,\ldots,k.$
$s = (k+1),(k+2),\ldots,n.$

$A_{ds} = a_{ds}$ $\quad d = (k+1),(k+2),\ldots,m.$
$s = (k+1),(k+2),\ldots,n.$

$A_{rf} = a_{rf}$ $\quad r = 1,2,\ldots,k.$ (square identity matrix)
$f = (n+1),(n+2),\ldots,(n+k).$

$A_{rh} = a_{rh}$ $\quad r = 1,2,\ldots,k.$ (null matrix)
$h = (n+k+1),(n+k+2),\ldots,(n+m).$

$A_{df} = a_{df}$ $\quad d = (k+1) + (k+2),\ldots,m.$ (null matrix)
$f = (n+1),(n+2),\ldots,(n+k).$

$A_{dh} = a_{dh}$ $\quad d = (k+1),(k+2),\ldots,m.$ (identity matrix)
$h = (n+k+1),(n+k+2),\ldots,(n+m).$

---

* The idea behind this form of decomposition will be clear soon. See Müller-Merbach, H.,"Lineare Plannungsrechnung mit Parametrisch veränderten Koeffizienten der Bedingung Matrix", in Ablauf-undPlannungsforschung 8(1967),H.1, pp. 33-39

Having established an initial basis $x_{n+1}$ , $x_{n+2},\ldots,x_{n+m}$, we may perform, say, the simplex method to develop the objective function.
Now, assume that the first k real variables $x_1, x_2 \ldots, x_k$ . replaced the first k slack variables $x_{n+1} x_{n+2},\ldots,x_{n+k}$ . This can be done without loss of generality since the numbering of the vectors can be done arbitrarily. Our tableau at the end of the Kth iteration would be as follows:

| | $x_r$ | $x_s$ | $x_f$ | $x_h$ | |
|---|---|---|---|---|---|
| $x_r$ | $Y_{rr}$ | $Y_{rs}$ | $Y_{rf}$ | $Y_{rh}$ | $\bar{P}_r^o$ |
| $x_h$ | $Y_{dr}$ | $Y_{ds}$ | $Y_{df}$ | $Y_{dh}$ | $\bar{P}_d^o$ |
| | $\beta_r$ | $\beta_s$ | $\beta_f$ | $\beta_h$ | |

Where:

$Y_{rr} = I$ an identity matrix.

$Y_{dr} = O$ a null matrix.

$Y_{rs} = A_{rr}^{-1} \cdot A_{rs}$ (§)

$$Y_{ds} = A_{ds} - A_{dr} \cdot A_{rr}^{-1} \cdot A_{rs}$$

$$= A_{ds} - A_{dr} \cdot Y_{rs}$$

$$= A_{ds} + Y_{df} \cdot A_{rs} \qquad \text{since } Y_{df} = - A_{dr} \cdot A_{rr}^{-1}$$

(§) What we are doing is "block-pivoting" on $A_{rr}$, see Dantzig, G., Linear Programming and Extensions, op. cit. pp. (232-234)

$$Y_{rf} = A_{rr}^{-1}$$

$$Y_{df} = - A_{dr} \cdot A_{rr}^{-1}$$

$$= -A_{dr} \cdot Y_{rf}$$

$Y_{rh} = O$ a null matrix

$Y_{dh} = I$ an identity matrix.

$$\overline{P}_r^o = A_{rr}^{-1} \cdot P_r^o$$

$$\overline{P}_d^o = P_d^o - A_{dr} A_{rr}^{-1} P_r^o$$

$$= P_d^o - A_{dr} \overline{P}_r^o$$

$$\beta_r = O$$

$$\beta_s = C_r \cdot A_{rr}^{-1} A_{rs} - C_s \quad (\S)$$

$$= C_r \cdot Y_{rs} - C_s$$

$$= Z - C$$

$$\beta_f = C_r \cdot A_{rr}^{-1}$$

$$\beta_h = O$$

$$f(x) = C_r \cdot A_{rr}^{-1} P_r^o$$

$$= C_r \cdot \overline{P}_r^o \qquad \text{since } \overline{P}_r^o = A_{rr}^{-1} P_r^o$$

---

(§) Since the problem is a maximization one, it is assumed that the objective function row vector is multiplied with -1.

The Simplex tableaux at the end of the Kth iteration

| | $X_r$ | $X_s$ | $X_f$ | $X_h$ | |
|---|---|---|---|---|---|
| $X_r$ | $(Y_{rr})$<br>$I$ | $(Y_{rs})$<br>$A_{rr}^{-1} \cdot A_{rs}$ | $(Y_{rf})$<br>$A_{rr}^{-1}$ | $(Y_{rh})$<br>$0$ | $\bar{P}_r^o$<br>$(A_{rr}^{-1} \cdot P_r^o)$ |
| $X_n$ | $(Y_{dr})$<br>$0$ | $(Y_{ds})$<br>$A_{ds}-A_{dr}.A_{rr}^{-1}.A_{rs}$<br>or<br>$(A_{ds}-A_{dr}.Y_{rs})$<br>or<br>$(A_{ds}+Y_{df}.A_{rs})$ | $(Y_{df})$<br>$-A_{dr}.A_{rr}^{-1}$ | $(Y_{dh})$<br>$I$ | $\bar{P}_d^o$<br>$(P_d^o - A_{dr}.A_{rr}^{-1} \cdot P_r^o)$<br>or<br>$(P_d^o - A_{dr}.\bar{P}_r^o)$ |
| | $(\beta_r)$<br>$(0)$ | $\beta_s$<br>$C_r.A_{rr}^{-1} A_{rs}-C_s$<br>or<br>$(C_r.Y_{rs}-C_s)$ | $\beta_f$<br>$C_r.A_{rr}^{-1}$ | $\beta_h$<br>$(0)$ | $f(X)= C_r \cdot A_{rr}^{-1} P_r^o$<br>$= C_r.\bar{P}_r^o$ |

It is interesting to notice that our tableau can be rewritten as the product of two matrices as follows:

$$\begin{pmatrix} A_{rr}^{-1} & 0 & 0 \\ -A_{dr} \cdot A_{rr}^{-1} & I & 0 \\ C_r \cdot A_{rr}^{-1} & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A_{rr} & A_{rs} & I & 0 & P_r^o \\ A_{dr} & A_{ds} & 0 & I & P_d^o \\ -C_r & -C_s & 0 & 0 & 0 \end{pmatrix}$$

Thus we see that each simplex tableau is the product of two matrices. The first matrix is essentially the inverse of the basis $B^{-1}$ considered by the RSM, and the second is nothing more than a complete tabulation of the original coefficients in the objective function and the system of the constraints.

To examine the possibility of introducing one of the nonbasic variables $x_s$ ; or in other words, to move towards an adjacent extreme point, we need to:

1. Define $\beta_s$ ; the relative cost coefficients of the nonbasic variables $x_s$ $s = k + 1 , k + 2,\ldots,n$. This could be done as performed by the RSM, through:

$$\beta_s = \begin{pmatrix} C_r \cdot A_{rr}^{-1} & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A_{rs} \\ A_{ds} \\ -C_s \end{pmatrix}$$

$$\beta_s = \left( C_r \cdot A_{rr}^{-1} \cdot A_{rs} - C_s \right)$$

(notice that $\beta_s$ is not affected with $A_{ds}$ since $A_{ds} \cdot 0 = 0$).

2. To determine the largest negative element of the vector $\beta_s$. Let it be $\beta_e$, where the index e indicates the newcoming variable $x_e$.

3. To define the vector $\begin{bmatrix} Y_{re} \\ Y_{de} \end{bmatrix}$ (§); the value of the newcoming variable $x_e$ in terms of the current basis. This could be done as usual by multiplying the column $\begin{bmatrix} a_{re} \\ a_{de} \end{bmatrix}$ (§§) $r + d = m$ , with the inverse of the basis

$$\begin{bmatrix} Y_{re} \\ Y_{de} \end{bmatrix} = \begin{bmatrix} A_{rr}^{-1} & 0 \\ -A_{dr} \cdot A_{rr}^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} a_{re} \\ a_{de} \end{bmatrix}$$

Then

$$Y_{re} = A_{rr}^{-1} \cdot a_{re} \qquad \text{and}$$

$$Y_{de} = a_{de} - A_{dr} \cdot A_{rr}^{-1} a_{re}$$
$$= a_{de} - A_{dr} \cdot Y_{re} \qquad \text{since } Y_{re} = A_{rr}^{-1} a_{re}$$

Such a conclusion is very important. It is the basic idea of the KM. To define the subvector $Y_{de}$, we do not need, any more, to define the submatrix $Y_{dr} = - A_{dr} \cdot A_{rr}^{-1}$ explicitly.

---

(§) It is the vector $Y_{ie}$; $r+d = m$
(§§) It is the vector $a_{ie}$; $r+d = m$

We need only:

First: To define the subvector $Y_{re}$

through

$$Y_{re} = A_{rr}^{-1} \cdot a_{re}$$

Second: To define the subvector $Y_{de}$

through

$$Y_{de} = a_{de} - A_{dr} \cdot Y_{re} \quad (\S)$$

So constituted, it should be fairly clear that the Keystone is not the inverse of the basis $B^{-1}$, as claimed by the RSM. It is, however, the inverse of the submatrix $A_{rr}$ as previously defined, since all the information needed to perform another move over the surface of the convex set can be derived directly when defining $A_{rr}^{-1}$.

## 3.4 GENERATING EXTREME POINT SOLUTIONS.

Consider the following LP problem

$$\begin{aligned} &\max && C'_r \cdot X_r \\ &\text{subject to} && A_{rr} \cdot X_r \leq P_r^o \\ &\text{and} && X_r \geq 0 \\ &\text{where} && r = 1,2,\ldots,k \end{aligned}$$

---

(§) The constant vector $\bar{P}_i^o$ is defined according to the same rule:

$$\bar{P}_r^o = A_{rr}^{-1} \cdot P_r^o$$

$$\bar{P}_d^o = P_d^o - A_{dr}\bar{P}_r^o$$

Applying, say, the simplex method, let the optimal solution obtained at the Kth. iteration be as follows:

$$\bar{X}_r = \bar{P}^o_r = A^{-1}_{rr} \cdot P^o_r$$

and the relative cost coefficients

$$\beta_f = C_r \cdot A^{-1}_{rr}$$

where $A^{-1}_{rr}$ is the inverse of the current basis.

Now suppose that the matrix $A_{rr}$ is enlarged with (m-k) constraints and (n-k) activities. The new matrix becomes of $A_{ij}$ dimensions i = 1,2,...,m j = 1,2,...,n.

Let us assume further that our solution $X'_r$ holds, as well, for the (m-k) constraints. The question now is, how could we further work on the problem $A_{ij}$ without resolving the problem $A_{rr}$ from the beginning ? (1)

## Applying the standard Simplex Method

To be able to move to an adjacent extreme point when using the simplex method, we must build up the whole simplex tableau, Tsim:

| | | | | |
|---|---|---|---|---|
| $x_r$ | I | $Y_{rs}$ | $Y_{rf}$ | O | $\bar{P}^o_r$ |
| $x_h$ | O | $Y_{ds}$ | $Y_{df}$ | I | $\bar{P}^o_d$ |
| | O | $\beta_s$ | $\beta$ | O | $C_r \cdot \bar{P}^o_r$ |

1) Bloech, J., "Zum Problem der nachträglichen Änderung industrieller Produktionsprogramme", in: Zeitschrift für Betriebswirtschaft, Bd. 36 (1966), S. 186 - 197

Recalling that the simplex tableau Tsim is

Simplex tableau = inverse tableau x original tableau

$$\text{Tsim} = \text{Tinv. Torg}$$

$$\text{where Tinv} = \begin{pmatrix} A_{rr}^{-1} & 0 & 0 \\ -A_{dr} \cdot A_{rr}^{-1} & I & 0 \\ C_r \cdot A_{rr}^{-1} & 0 & 1 \end{pmatrix}$$

and Torg

$$\begin{pmatrix} A_{rr} & A_{rs} & I & 0 & P_r^o \\ A_{dr} & A_{ds} & 0 & I & P_d^o \\ -C_r & -C_s & 0 & 0 & 0 \end{pmatrix}$$

and since the inverse of the current basis $A_{rr}^{-1}$ and the relative cost coefficients $C_r \cdot A_{rr}^{-1}$ are already defined, we need only to define the submatrix $-A_{dr} \cdot A_{rr}^{-1}$

$d = (k+1),(k+2),\ldots,m$ to complete Tinv. Multiplying Tinv by Torg we obtain the required Tsim.

$$\begin{array}{|cccc|c} I & Y_{rs} & Y_{rf} & 0 & \bar{P}_r^o \\ 0 & Y_{ds} & Y_{df} & I & \bar{P}_r^o \\ \hline 0 & \beta_s & \beta_f & 0 & C_r \cdot A_{rr}^{-1} P_r^o \end{array}$$

let $\beta_e$ = max $\beta$ where the index e indicates the newcoming variable $X_e$, and let

$\bar{P}_g^o / y_{ge}$ be the min-nonnegative of

$\bar{P}_i^o \Big/ y_{ie}$ where i = 1,2,...,m and the index g indicates the outgoing row. Performing the usual transformation procedures pivoting on the element $y_{ge}$ we obtain the simplex tableau Tsim at the (k+1)th iteration.

Applying the Revised Simplex Method.

Applying the revised simplex method, we need only to complete Tinv

$$\begin{bmatrix} A_{rr}^{-1} & 0 \\ -A_{dr} \cdot A_{rr}^{-1} & I \end{bmatrix} \begin{bmatrix} \bar{P}_r^o \\ \bar{P}_d^o \end{bmatrix}$$

$$\begin{bmatrix} C_r A_{rr}^{-1} & 0 \end{bmatrix} \quad C_r \cdot A_{rr}^{-1} \; P_r^o$$

Having defined the inverse of the "current" solution $A_{rr}^{-1}$, the relative cost coefficients $C_r \cdot A_{rr}^{-1}$ . and the subvector $\bar{P}_r^o$, we need only to define

the submatrix $- A_{dr} \cdot A_{rr}^{-1}$ and the sub-vector-column $\bar{P}_d^o$ (1) .

---

1) It is defined through the formula

$$\bar{P}_d^o = P_d^o - A_{dr} \cdot A_{rr}^{-1} \cdot P_r^o$$

$$= P_d^o - A_{dr} \cdot \bar{P}_r^o$$

(See page 43)

Building up the Tinv, we may compute $\beta_j$

$$\beta_j = C_r \cdot A_{rr}^{-1} \cdot A_{rj} - C_j \quad (\S) \quad j = 1,2,\ldots,n$$

Let $\beta_e = \max|\beta_j|$ where the index e indicates the newcoming variable $X_e$. Determining the value of the newcoming variable $X_e$ in terms of the current basis through:

$$Y_{ie} = \text{Tinv} \cdot a_{ie} \qquad i = 1,2,\ldots,m$$

$$\begin{bmatrix} Y_{re} \\ Y_{de} \end{bmatrix} = \begin{bmatrix} A_{rr}^{-1} & 0 \\ -A_{dr} \cdot A_{rr}^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} a_{re} \\ a_{de} \end{bmatrix}$$

and performing the usual transformation procedure pivoting on the element $Y_{ge}$ (1), we obtain the Tinv at the (k+1)th iteration.

---

1) It is assumed that $\bar{P}_g^o / Y_{ge}$ is the min-nonnegative of $\bar{P}_i^o / Y_{ie}$ $\quad i = 1,2,\ldots,m$

§ Applying the RSM, $\beta_j$, is obtained generally through:

$$\beta_j = \beta_i \cdot A_{ij} - C_j$$

but since $\beta_h$ $\quad h = k+1, k+2,\ldots,m$ is null, it is enough to consider only $\beta_r \cdot A_{rj}$. And once $\beta_r$ is already defined, we need only to consider $\beta_s$ which is defined through:

$$\beta_s = \beta_f \cdot A_{rs} - C_s$$

where s; $s = k+1, K+2, \ldots, n$ indicates the real nonbasic variables.

or $\quad = C_r \cdot A_{rr}^{-1} \cdot A_{rs} - C_s$

since $\quad \beta_f = C_r A_{rr}^{-1}$

Applying KAIRO METHOD :

An imaginary tableau for the enlarged problem would be as follows:

| | $X_r$ | $X_s$ | $b_r$ | $b_d$ | |
|---|---|---|---|---|---|
| $X_r$ | I | $A_{rs}$ | $A_{rr}^{-1}$ | O | $\bar{P}^{o}_{r}$ |
| $b_d$ | O | $A_{ds}$ | ? | I | $\bar{P}^{o}_{d}$ |
| | O | $C_s$ | $\beta_r$ (§) | O | |

To move to an adjacent extreme point, we must first define $P'^{o}_{d}$ i.e., the redundancy at $b_d$ $d = k+1,\ldots,m$.

This is defined through

$$\bar{P}^{c}_{d} = P^{o}_{d} - A_{dr} \cdot A_{rr}^{-1} \cdot \bar{P}^{o}_{r}$$

$$= P^{o}_{d} - A_{dr} \cdot \bar{P}^{o}_{r}$$

since $\bar{P}^{o}_{r} = A_{rr}^{-1} \cdot P^{o}_{r}$

It is clear that $\bar{P}^{o}_{d}$ $d = n+1,\ldots,m$ must be nonnegative, otherwise the solution would be nonfeasible. (1)

---

(§) From now on, $\beta_r$ will indicate the cost coefficients of the basic variables defined through the relation:

$$\beta_r = C_r \cdot A_{rr}^{-1}$$

(1) Müller-Merbach, Operation Research, op cit. p. 106

The inverse of the basis would be as follows

| | $b_r$ | $b_d$ | |
|---|---|---|---|
| $X_r$ | $A_{rr}^{-1}$ | 0 | $\bar{P}_r^o$ |
| $b_d$ | ? | I | $\bar{P}_d^o$ |
| | $\beta_r$ | 0 | |

where $\beta_r = C_r \cdot A_{rr}^{-1}$

The basic achievement of the KM is to prove that we are able to generate a chain of nonnegative vertex solutions without completing the inverse of the basis. Or in other words, without defining the submatrix

$$- A_{dr} \cdot A_{rr}^{-1}$$

explicitly as required by the RSM.

Generally, to move towards an adjacent extreme point, the following steps must be applied:

Step 1. Defining the relative cost coefficients $\beta_s$ $s=K+1,K+2,\ldots,n$ . This is accomplished through

| | $b_r$ | $b_d$ | $X_s$ |
|---|---|---|---|
| $X_r$ | $A_{rr}^{-1}$ | 0 | $\rightarrow a_{1s} \rightarrow$ |
| $b_d$ | | I | $a_{2s} \rightarrow$ |
| | | | $a_{Ks} \rightarrow$ |
| | $\bar{\beta}_1 \bar{\beta}_2 \ldots \bar{\beta}_K$ | $\ldots 0 \ldots 0$ | |

$$\beta_s = (\beta_1 , \beta_2, \ldots, \beta_K) \cdot \begin{pmatrix} a_{1s} \\ a_{2s} \\ \vdots \\ a_{Ks} \end{pmatrix} - C_s$$

It is interesting here to notice that $\beta_s$ could be defined in two ways:

i) $\beta_s = C_r \cdot Y_{rs} - C_s$

where $Y_{rs}$ is the value of the newcoming variable in terms of the basis. $Y_{rs}$ is defined through:

$$Y_{rs} = A_{rr}^{-1} \cdot A_{rs}$$

or

ii) $$\beta_s = \beta_r \cdot A_{rs} - C_s$$

since $\beta_r = C_r \cdot A_{rr}^{-1}$

It might be clear that although the mathematical argument in the two approaches is the same, that is, we want to define the following expression:

$$\beta_s = C_r \cdot A_{rr}^{-1} \cdot A_{rs} - C_s$$

The first approach requires the multiplication of

a) two matrices, namely,

$$A_{rr}^{-1} \cdot A_{rs}$$

to obtain $Y_{rs}$ , then

b) vector by matrix, namely

$$C_r \cdot Y_{rs}$$

whearas the second approach requires the multiplication of

a) vector by matrix, namely

$$C_r \cdot A_{rr}^{-1}$$

to obtain $\beta_r$ , then

b) vector by matrix, namely

$$\beta_r \cdot A_{rs}$$

However, since $\beta_r$ is already defined we may adapt the second approach

Step 2. Having defined $\beta_s$ i.e., the relative cost coefficients of the nonbasic variables $x_s$ $s = K+1, K+2, \ldots, n$, let $\beta_e$ be the largest negative element of $\beta_s$ . This means that the variable $x_e$ is the newcoming variable.

Step 3. Determining the vector $Y_{ie}$ $i = 1,2,\ldots,m$ i.e., the value of the newcoming variable in terms of the current basis. This is done by the RSM through

$$Y_{ie} = \begin{pmatrix} Y_{re} \\ \\ Y_{de} \end{pmatrix} = B^{-1} \cdot a_{ie} \qquad i=1,2,\ldots,m$$

$$= \begin{pmatrix} A_{rr}^{-1} & & O \\ \\ -A_{dr} \cdot A_{rr}^{-1} & & I \end{pmatrix} \cdot \begin{pmatrix} a_{re} \\ \\ a_{de} \end{pmatrix}$$

$$\left|\begin{array}{c} Y_{re} \\ \hline Y_{de} \end{array}\right| = \left|\begin{array}{c} A_{rr}^{-1} \cdot a_{re} \\ \hline a_{de} - A_{dr} \cdot A_{rr}^{-1} a_{re} \end{array}\right|$$

Such a step necessitates the completion of the inverse of the basis $B^{-1}$.

Now let us see how this step could be done without defining the submatrix $- A_{dr} \cdot A_{rr}^{-1}$ explicitly.

We have seen that

$$Y_{de} = a_{de} - A_{dr} \cdot A_{rr}^{-1} a_{re}$$

but since $Y_{re} = A_{rr}^{-1} \cdot a_{re}$

the expression could be written

$$Y_{de} = a_{de} - A_{dr} \cdot Y_{re}$$

which is a relation between the subvector $Y_{re}$ which is defined at any iteration once $A_{rr}^{-1}$ is defined. Furthermore, the subvector $a_{de}$ and the submatrix $A_{dr}$. Both of them are elements of the original tableau.

Step 4. To define the pivot row, we determine, as usual, the minimum nonnegative of the ratio

$$\bar{P}_i^o / Y_{ie}$$

let it be $\bar{P}_g^o / Y_{ge}$, which means that the constraint g is going to be satisfied.

Step 5. Having defined the variable to enter the basis and the variable to leave the basis, the inverse of the new basis $A_{ww}^{-1}$ determined through inverting the following system:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots a_{1K} & a_{1e} \\ a_{21} & a_{22} & \cdots a_{2K} & a_{2e} \\ a_{K1} & a_{K2} & \cdots a_{KK} & a_{Ke} \\ a_{g1} & a_{g2} & \cdots a_{gK} & a_{ge} \end{pmatrix}$$

Determining the inverse $A_{ww}^{-1}$, we are able to define $P''^{o}_{i}$ (§) and the relative cost coefficients $\beta_w$, before trying another move starting from step 1.

Computationally, the vector $P''^{o}_{i}$ could be determined in two ways.

1. after defining the inverse $A_{ww}^{-1}$, the vector $P''^{o}_{i}$ is defined through:

$$\begin{pmatrix} P''^{o}_{w} \\ P''^{o}_{v} \end{pmatrix} = \begin{pmatrix} A_{ww}^{-1} \cdot \bar{P}^{o}_{w} & & w = 1,2,\ldots,K+1 \\ \bar{P}^{o}_{v} - A_{vw} \cdot P''^{o}_{w} & & v = K+2,\ldots,m \end{pmatrix}$$

$$w + v = m.$$

or

2. Independently from the inverse $A_{ww}^{-1}$ through the usual transformation process

$$P''^{o}_{g} = \bar{P}^{o}_{g} / Y_{ge}$$

$$P''_{i} = \bar{P}^{o}_{i} - Y_{ie} \cdot P''^{o}_{g} \qquad i \neq g$$

---

(§) $P''^{o}_{i}$ is the transformed value of $\bar{P}^{o}_{i}$.

## 3.5 SUMMARY.

Let us now piece together the discussion in the foregoing section and present an integrated view of the procedure as employed in the solution of a linear programming problem.

We assume that an extreme point solution in terms of k vectors of the original set is already defined. To move towards an adjacent extreme point we apply the following steps:

i) Determining the relative cost coefficients

$$\beta_s \quad s = (k+1),(k+2),\ldots,n \quad \text{through}$$

$$\beta_s = \beta_r \cdot A_{rs} - C_s$$

two situtations may arise

1) All computed ß are zero or of positive value, which means that no further improvement in the objective function can be expected. The last feasible solution is an optimal solution.

2) some of ß are negative. In this case we select the vector with the most negative ß to be our newcoming vector. Let it $x_e$

ii) Determining the value of the newcoming variable in terms of the current basis through

$$Y_{re} = A_{rr}^{-1} \cdot a_{re}$$

$$Y_{de} = a_{de} - A_{dr} Y_{re}$$

iii) Determining the outgoing vector through finding min nonnegative value of the ratio

$$\bar{P}_i^o / Y_{ie} \qquad i = 1,2,\ldots,m$$

let it $\bar{P}_g^o / Y_{ge}$

iiii) Finally, we determine the inverse of the current basis, including the newcoming and the outgoing vectors "e" and "g" respectively, by inverting the following system:

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1k} & a_{1e} \\ a_{21} & a_{22} & \cdots & a_{2k} & a_{2e} \\ \cdots & \cdots & & \cdots & \\ a_{k1} & a_{k2} & \cdots & a_{kk} & a_{ke} \\ a_{g1} & a_{g2} & \cdots & a_{gk} & a_{ge} \end{matrix}$$

Having defined the new inverse, we are able to examine the possibility of introducing another variable starting from step (i).

## 3.6 The KM computationally.

We must at the outset of our discussion stress the fact that the new method is not suitable for hand computation. The following discussion will show clearly how the difficulties arise during the course of the iterative process. For this reason, we will not try to develop a standard tableau. Our concern will be devoted, basically, to show how the successive extreme points are generated.

We have proved, however, that having defined the inverse of the basis $A_{rr}^{-1}$ and the associated relative cost coefficients $\beta_r$, a new extreme point could be defined through applying the following steps.

1. Defining the relative cost coefficients of the non-basic variables $X_s$ through:

$$\beta_s = \beta_r \cdot A_{rs} - C_s$$

$$r = 1,2,\ldots,k$$
$$s = k+1,k+2,\ldots,n$$

where k implies the number of the iterations performed; or more precisely the number of the real variables introduced to the basis and the number of the constraints which have been completely exhausted.

Through our theoretical presentation, we have assumed that the first k slack variables "may replace the first slack variables". Or in other words, the first k constraints have been completely exhausted by introducing the first k real variables.

Practically, this must not be always the case. For instance, the $J^{th}$ variable may replace the $i^{th}$ constraint where $j \neq i$

Another point is that, while the inverse $A_{rr}^{-1}$ represents the inversion of the coefficients

$$a_{rr} \qquad r = 1,2,\ldots,k.$$

We may find, after performing k iterations, that any k variables have replaced any k constraints. It means that the symbol $A_{rr}$ is not suitable anymore. Another symbol with two different indices must be choosed; $A_{zq}$, for instance, where z indicates the constraints which have been exhausted. z might be as follows:

$$z = 5,\ 3,\ 2\ .$$

and q indicates the real variables introduced to the basis. q might be as follows:

$$q = 4,\ 1,\ 3\ .$$

This means that, at the first iteration, the variable $x_4$ replaced the constraint $b_5$, At the second iteration, the variable $x_1$ replaced the constraint $b_3$, and at the third iteration the variable $x_3$ replaced the constraint $b_2$. This of course when the dimensions of the basis are 3.3 (§)

In such a case, the coefficients to be inverted are

$$A_{zq} = \begin{Bmatrix} a_{54} & a_{51} & a_{53} \\ a_{34} & a_{31} & a_{33} \\ a_{24} & a_{21} & a_{23} \end{Bmatrix}$$

---

(§) The number of the iterations performed could be more than 3.

the inverse $A_{zq}^{-1}$ would be

$$\begin{pmatrix} y_{54} & y_{51} & y_{53} \\ y_{34} & y_{31} & y_{33} \\ y_{24} & y_{21} & y_{23} \end{pmatrix}$$

The relative cost coefficients associated to the basis are defined through

$$\beta_z = c_q \cdot A_{zq}^{-1}$$

$$= \begin{bmatrix} c_4 & c_1 & c_3 \end{bmatrix} \cdot \begin{pmatrix} y_{54} & y_{51} & y_{53} \\ y_{34} & y_{31} & y_{33} \\ y_{24} & y_{21} & y_{23} \end{pmatrix}$$

$$= \begin{bmatrix} \beta_5 & \beta_3 & \beta_2 \end{bmatrix}$$

To determine the value of the nonbasic variables $x_s$ in terms of the basis $A_{zq}^{-1}$ we apply :

$$\beta_s = \beta_z \cdot a_{zs} - c_s$$

where $s = 2,5,6,\ldots,n$

$$\beta_s = \begin{bmatrix} \beta_5 & \beta_3 & \beta_2 \end{bmatrix} \cdot \begin{pmatrix} a_{5s} \\ a_{3s} \\ a2s \end{pmatrix} - c_s$$

Finding the largest negative $\beta_s$, let it be $\beta_6$, we are ready to apply the second step.

2. Determining the value of the newcoming vector in terms of the inverse $A_{zq}^{-1}$ . This is done in two steps

   a) First we find $y_{z6}$ $\quad z = 5,3,2$ .

   through

$$y_{z6} = A_{zq}^{-1} \cdot a_{z6}$$

   then

   b) Finding $y_{d6}$ $\quad d = 1,4,6,\ldots,m$

$$y_{d6} = A_{d6} - A_{dq} \cdot y_{z6}$$

   (notice that the dimension of q will be always equal to the dimension of z, even if the solution is degenerated i.e., one or more of $p'^{o}_{z}$ are equal to zero.)

Having defined $y_{i6}$ $\quad i = 1,2,\ldots,m$, the pivot element $(y_{g6})$ must be defined through:

$$\bar{p}^{\,o}_{g} / y_{g6} = \min \text{ nonnegtive } \bar{p}^{\,o}_{i} / y_{i6}$$

where the index g means that, let us say,(§) the constraint g is going to be satisfied.

Before applying the next step, we may define the new value of the vector $p^{o}_{i}$ through the normal transformation process:

---

(§) The index g may indicate one of the introduced real variables.

$$p''^{o}_{g} = \bar{p}^{o}_{g} / y_{g6}$$

$$p''^{o}_{i} = \bar{p}^{o}_{i} - y_{i6} \cdot p''^{o}_{g} \qquad i \neq g$$

let g = 7

3. Finally, we find the inverse of the current basis through inverting the following system:

$$\begin{pmatrix} a_{54} & a_{51} & a_{53} & a_{56} \\ a_{34} & a_{31} & a_{33} & a_{36} \\ a_{24} & a_{21} & a_{23} & a_{26} \\ a_{74} & a_{71} & a_{73} & a_{76} \end{pmatrix}$$

where the last column $a_{z6}$ the last row $a_{7q}$ and the pivot element $a_{76}$ have been annexed to the old system

$$\begin{pmatrix} a_{54} & a_{51} & a_{53} \\ a_{34} & a_{31} & a_{33} \\ a_{24} & a_{21} & a_{23} \end{pmatrix}$$

Let $A^{-1}_{zq}$ be the inverse of the current enlarged basis where

$$z = 5,3,2, \text{ and } 7$$

$$q = 4,1,3, \text{ and } 6$$

Before ending this iteration we may define $\beta_z$ z =5,3,2, and 7 through:

$$\beta_z = c_q \cdot A_{zq}^{-1}$$

q = 4,1,3, and 6

z = 5,3,2, and 7

To determine whether our solution is optimal, $\beta_s$ is defined through

$$\beta_s = \beta_z \cdot a_{zs} - c_s$$

where z = 5,3,2, and 7

and s = 2,5,7,8,...n

where two situations may arise

1. all of $\beta_s$ are positive, then our solution

$$\bar{x} = \begin{Bmatrix} \bar{x}_4 \\ \bar{x}_1 \\ \bar{x}_3 \\ \bar{x}_6 \end{Bmatrix} = \begin{Bmatrix} p'^{o}_5 \\ p'^{o}_3 \\ p'^{o}_2 \\ p'^{o}_7 \end{Bmatrix}$$

is optimal, while $b_d$ d = 1,4,5,7,8,..., M implies the redundant capacities

$$b_d = p'^{o}_d$$

2. One or more of the vector $\beta_z$ is negative, another iteration is required starting with step 2.

## 3.7 Illustrative example.

In order to show the computational difference between the simplex method, the revised simplex method and the KAIRO METHOD, it is desirable to work out a numerical example. During the course of iteration, the example will be solved, as well, using the standard simplex method and its revised form.

Consider the following LP problem.

$$9\,x_1 + 7\,x_2 + 8\,x_3 + 2\,x_4 + 1{,}5\,x_5$$

subject to

$$10\,x_1 + 5\,x_2 + 5\,x_3 + 9\,x_4 + 5\,x_5 \leq 65$$

$$6\,x_1 + 6\,x_2 + 8\,x_3 + 8\,x_4 + 6\,x_5 \leq 60$$

$$4{,}5\,x_1 + 18\,x_2 + 9\,x_3 + 6\,x_4 + 8\,x_5 \leq 81$$

$$3\,x_1 + 2\,x_2 + 1{,}5\,x_3 + 1\,x_4 + 1{,}5\,x_5 \leq 30$$

$$1{,}5\,x_1 + 3\,x_2 + 4\,x_3 + 2\,x_4 + 1{,}25\,x_5 \leq 29$$

and

$$x_1\,,\ x_2,\ldots,x_5 \geq 0$$

Inserting the required slack variables, the above system of inequalities can be transformed into a system of equations. A matrix, which depicts the starting point of the computation, can be now drawn:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $b_1$ | 10 | 5 | 5 | 9 | 5 | 1 | 0 | 0 | 0 | 0 | 65 |
| $b_2$ | 6 | 6 | 8 | 8 | 6 | 0 | 1 | 0 | 0 | 0 | 60 |
| $b_3$ | 4,5 | 18 | 9 | 6 | 8 | 0 | 0 | 1 | 0 | 0 | 81 |
| $b_4$ | 3 | 2 | 1,5 | 1 | 1,5 | 0 | 0 | 0 | 1 | 0 | 30 |
| $b_5$ | 1,5 | 3 | 4 | 2 | 1,25 | 0 | 0 | 0 | 0 | 1 | 29 |
| | -9 | -7 | -8 | -2 | -1,5 | 0 | 0 | 0 | 0 | 0 | |

The first iteration

a) using the standard SM

since $c_1$ is the largest negative $c_j$ and the ratio $p_1^o/a_{11}$ is the min. nonnegative of $p_i^{'}/a_{i1}$, we perform the normal transformation process pivoting on $a_{11}$. The simplex tableau at the end of the first iteration would be:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0,5 | 0,5 | 0,9 | ,5 | 0,1 | 0 | 0 | 0 | 0 | 6,5 |
| $b_2$ | 0 | 3 | 5 | 2,6 | 3 | -0,6 | 1 | 0 | 0 | 0 | 21 |
| $b_3$ | 0 | 15,75 | 6,75 | 1,95 | 5,75 | -0,45 | 0 | 1 | 0 | 0 | 51,75 |
| $b_4$ | 0 | 0,5 | 0 | -1,7 | 0 | -0,3 | 0 | 0 | 1 | 0 | 10,5 |
| $b_5$ | 0 | 2,25 | 3,25 | 0,65 | 0,5 | -0,15 | 0 | 0 | 0 | 1 | 19,25 |
| | 0 | -2,5 | -3,5 | 6,1 | 3 | 0,9 | 0 | 0 | 0 | 0 | 58,5 |

$f(x) = 58,5$

b) Using the revised simplex method.

Determining that $x_1$ replaces $b_1$, the inverse of the basis $B^{-1}$ after pivoting on $a_{11}$ would be as follows:

| | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | |
|---|---|---|---|---|---|---|
| $x_1$ | 0,1 | | | | | 6,5 |
| $b_2$ | -0,6 | 1 | | | | 21 |
| $b_3$ | -0,45 | | 1 | | | 51,75 |
| $b_4$ | -0,3 | | | 1 | | 10,5 |
| $b_5$ | -0,15 | | | | 1 | 19,25 |
| | 0,9 | 0 | 0 | 0 | 0 | |

$f(x) = 58,5$

c) Using Kairo method:

At the first iteration, no extra computation is required. We define the largest negatvie $c_j$ ; $c_1$ in our example - and the min. nonnegative of the ratio

$p_i^o / a_{i1}$ ; $p_1^o / a_{11}$ in our case

The the inverse $A_{zq}^{-1}$ is

$$A_{zq}^{-1} = {}^{1}/_{a_{11}} = 0,1 \qquad z = q = 1$$

$$\beta_z = c_z \cdot A_{zq}^{-1}$$

$$= 9 \cdot ,1$$

$$= 0,9$$

The constant column vector, $p^o$, is defined through the normal transformation process

$$p'^{o}_{1} = p_1^o / a_{11} = ,$$

$$p'^{o}_{d} = p^o - y_{de} \cdot p'^{o}_{1} \qquad d = 2,3,4,5 \quad q = 1$$

$$\begin{pmatrix} p'^{o}_{2} \\ p'^{o}_{3} \\ p'^{o}_{4} \\ p'^{o}_{5} \end{pmatrix} = \begin{pmatrix} 60 \\ 81 \\ 30 \\ 29 \end{pmatrix} - \begin{pmatrix} 6 \\ 4.5 \\ 3 \\ 1,5 \end{pmatrix} \cdot 6,5 = \begin{pmatrix} 21 \\ 51,75 \\ 10,5 \\ 19,25 \end{pmatrix}$$

($y_{de}$ at the first iteration is equal to $a_{de}$. The index e implies the new coming variable.

The value of the objective function is

$$f(x) = c_q \cdot p'^{o}_{q} \qquad q = 1$$

$$= \cdot 6,5$$

$$= 58,5$$

(c) The second iteration

The largest negative $c'_j$ is $c'_3$, the min nonnegative ratio $p'^{o}_{i} / y_{i3}$ is $p'^{o}_{2} / y_{23}$.

Pivoting on $y_{23}$, the simplex tableau at the end of the second iteration is

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0,2 | 0 | 0,64 | 0,2 | 0,16 | -0,1 | 0 | 0 | 0 | 4,4 |
| $x_3$ | 0 | 0,6 | 1 | 0,52 | 0,6 | -,12 | -0,2 | 0 | 0 | 0 | 4,2 |
| $b_3$ | 0 | 11,7 | 0 | -1,56 | 1,7 | ,36 | -1,35 | 1 | 0 | 0 | 23,40 |
| $b_4$ | 0 | 0,5 | 0 | -1,7 | 0 | -0,3 | 0 | 0 | 1 | 0 | 10,5 |
| $b_5$ | 0 | 0,3 | 0 | -1,04 | -1,45 | 0,24 | -,65 | 0 | 0 | 1 | 5,6 |
| | 0 | -0,4 | 0 | 7,92 | 5,1 | 4,8 | 0,7 | 0 | 0 | 0 | 73,2 |

$f(x) = 73,2$

b) The second iteration produces the MSV.

To move to an adjacent extreme point, $\beta_s$ must be defined through

$$\beta_i = \begin{bmatrix} \beta_1 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot a_{ij} - c_i \quad (*)$$

$$\beta_1 = \begin{bmatrix} 0,9 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 10 \\ 6 \\ 4,5 \\ 3 \\ 1,5 \end{pmatrix} - 9 = 0$$

$$\beta_2 = \begin{bmatrix} ,9 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 5 \\ 6 \\ 18 \\ 7 \\ 3 \end{pmatrix} - 7 = -2,5$$

$$\beta_3 = \begin{bmatrix} ,9 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 5 \\ 8 \\ 9 \\ 1,5 \\ 4 \end{pmatrix} - 8 = -3,5$$

(*) The vector $(0,9 \quad 0 \quad 0 \quad 0 \quad 0)$ is the relative cost coefficients appears as the last row in $B^{-1}$

$$\beta_4 = \begin{bmatrix} 0{,}9 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 9 \\ 8 \\ 6 \\ 1 \\ 2 \end{Bmatrix} - 2 = 6{,}1$$

$$\beta_5 = \begin{bmatrix} 0{,}9 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 5 \\ 6 \\ 8 \\ 1{,}5 \\ 1{,}25 \end{Bmatrix} - 1{,}5 = 3$$

since $\beta_3$ is the largest negative $\beta_i$ , then $x_3$ is the newcoming variable.

The value of $x_3$ in terms of the basis is defined through:

$$y_{i3} = B_{\cdot}^{-1} \cdot a_{i3}$$

$$\begin{matrix} y_{13} \\ y_{23} \\ y_{33} \\ y_{43} \\ y_{53} \end{matrix} = \begin{Bmatrix} 0{,}1 & & & & \\ -0{,}6 & 1 & & & \\ -0{,}45 & & 1 & & \\ -0{,}3 & & & 1 & \\ -0{,}15 & & & & 1 \end{Bmatrix} \cdot \begin{Bmatrix} 5 \\ 8 \\ 9 \\ 1{,}5 \\ 4 \end{Bmatrix} \begin{Bmatrix} . \\ . \\ . \\ . \\ . \end{Bmatrix}$$

The information acquired in a tableau form would be:

| | | | | | $\bar{x}_3$ | $p'^{o}$ |
|---|---|---|---|---|---|---|
| $x_1$ | 0,1 | | | | 0,5 | 6,5 |
| $b_2$ | -0,6 | 1 | | | (5) | 2 |
| $b_3$ | -0,45 | | 1 | | 6,75 | 51,75 |
| $b_4$ | -0,3 | | | 1 | 0 | 10,5 |
| $b_5$ | -0,15 | | | | 3,25 | 19,25 |
| | 0,9 | 0 | 0 | 0 | -3,5 | 58,5 |

pivoting on $y_{23}$ , the inverse at the end of the second iteration is:

| | | | | | | $p'^{o}$ |
|---|---|---|---|---|---|---|
| $x_1$ | 0,16 | -,1 | | | | 4,4 |
| $x_3$ | -0,12 | 0,2 | | | | 4,2 |
| $b_3$ | -0,36 | -1,35 | 1 | | | 23,4 |
| $b_4$ | -0,3 | 0 | | 1 | | 10,5 |
| $b_5$ | -0,24 | -0,65 | | | 1 | 5,6 |
| | -,48 | 0,7 | 0 | 0 | 0 | 73,2 |

$f(x) = 73,2$

c) The second iteration

First we must define $\beta_s$ through:

$$\beta_s = \beta_z \cdot a_{zs} - c_s \qquad \begin{matrix} z = 1 \\ s = 2,3,4,5 \end{matrix}$$

$$\beta_2 = 0,9 \cdot 5 - 7 = -2,5$$

$$\beta_3 = 0,9 \cdot 5 - 8 = -3,5$$

$$\beta_4 = 0,9 \cdot 9 - 2 = 6,1$$

$$\beta_5 = 0,9 \cdot 5 - 1,5 = 3$$

The largest negative $\beta_s$ is $\beta_3$. The value of $x_3$ in terms of the basis is defined through:

1) $$y_{z3} = A^{-1}_{zq} \cdot a_{z3} \qquad \begin{matrix} z = 1 \\ q = 1 \end{matrix}$$

$$y_{13} = 0,1 \cdot 5$$

2) $$y_{d3} = a_{d3} - a_{dq} \cdot y_{z3} \qquad \begin{matrix} z = 1 \\ q = 1 \\ d = 2,3,4,5 \end{matrix}$$

$$\begin{pmatrix} y_{23} \\ y_{33} \\ y_{43} \\ y_{53} \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \\ 1,5 \\ 4 \end{pmatrix} - \begin{pmatrix} 6 \\ 4,5 \\ 3 \\ 1,5 \end{pmatrix} \cdot 0,5 \qquad \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

The min. nonnegative $p_i^o / y_{i3}$ is

$$p_g^o / y_{g3} = p_2^o / y_{23}$$

The constant column vector, $p^o$ is defined through the normal transformation process pivoting on $y_{23}$

$$p'^o_g = p_g^o / y_{ge}$$

$$p'^o_i = p_i^o - y_{ie} \cdot p'^o_g \qquad i \neq g$$

where e and g indicate the newcoming and the outgoing variables respectively.

$$p'^o_2 = p_2^o / y_{23}$$

$$= 21 / 5 = 4,2$$

$$\begin{pmatrix} p'^o_1 \\ p'^o_3 \\ p'^o_4 \\ p'^o_5 \end{pmatrix} = \begin{pmatrix} 6,5 \\ 51,75 \\ 10,5 \\ 19,25 \end{pmatrix} - \begin{pmatrix} 0,5 \\ 6,75 \\ 0 \\ 3,5 \end{pmatrix} \cdot 4,2 = \begin{pmatrix} 4,4 \\ 23,4 \\ 10,5 \\ 4,4 \end{pmatrix}$$

The inverse of the basis $A_{zq}^{-1}$ $\quad z = 1, \text{ and } 2$ $\quad q = 1, \text{ and } 3$

is defined through inverting the following system:

$$A_{zq} = \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix}$$

$$A_{zq} = \begin{pmatrix} 10 & 5 \\ 6 & 8 \end{pmatrix}$$

$$A_{zq}^{-1} = \begin{pmatrix} 0,16 & -0,1 \\ -0,12 & 0,2 \end{pmatrix}$$

$\beta_z$ is defined through: $\quad z = 1,2$ $\quad q = 1,3$

$$\begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix} = \begin{bmatrix} c_1 & c_3 \end{bmatrix} \cdot A_{zq}^{-1}$$

$$\begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix} = \begin{bmatrix} 9 & 8 \end{bmatrix} \begin{pmatrix} 1,6 & -,1 \\ -,12 & ,2 \end{pmatrix}$$

$$\begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix} = \begin{bmatrix} 0,48 & 0,7 \end{bmatrix}$$

The third iteration

Since $c'_2$ is the largest negative $c'_j$, $x_2$ will be introduced. The min. nonnegative of the ratio

$$p'^{o}_i \ / \ y_{i2} \quad \text{is} \quad p'^{o}_g \ / \ y_{g2}$$

Then $$p'^{o}_g \ / \ y_{g2} = p'^{o}_3 \ / \ y_{32} = 2$$

Pivoting on $y_{32}$, the simplex tableau at the third iteration will be:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $b_1$ | $b_2$ | $b_3$ (§) | $b_4$ | $b_5$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 0 | 0,65 | -,17 | 0,16 | -0,07 | -0,01 | 0 | 0 | 4 |
| $x_3$ | 0 | 0 | 1 | 0,6 | ,51 | -0,13 | 0,26 | -0,05 | 0 | 0 | 3 |
| $x_2$ | 0 | 1 | 0 | -0,13 | ,14 | ,02 | -0,11 | -0,08 | 0 | 0 | 2 |
| $b_4$ | 0 | 0 | 0 | -1,63 | -,07 | -0,31 | -0,05 | -0,04 | 1 | 0 | 9,5 |
| $b_5$ | 0 | 0 | 0 | 1 | 1,4 | 0,24 | -0,62 | -0,02 | 0 | 1 | 5 |
| | 0 | 0 | 0 | 7,86 | 5,15 | 0,49 | 0,65 | 0,03 | 0 | 0 | 74 |

The (x) = 74

The present solution is an optimal solution.

---

(§) The values are rounded.

b) Using the RSM.

First we define $\beta_s$ through:

$$\beta_j = \left(\beta_i \cdot a_{is}\right) - c_i$$

$$\beta_1 = \begin{pmatrix} 0{,}48 & 0{,}7 & o & o & o \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 6 \\ 4{,}5 \\ 3 \\ 1{,}5 \end{pmatrix} - 9 = 0$$

$$\beta_2 = \begin{pmatrix} 0{,}48 & 0{,}7 & o & o & o \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 6 \\ 18 \\ 2 \\ 3 \end{pmatrix} - 7 = -{,}4$$

$$\beta_3 = \begin{pmatrix} 0{,}48 & 0{,}7 & o & o & o \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 8 \\ 9 \\ 1{,}5 \\ 4 \end{pmatrix} - 8 = o$$

$$\beta_4 = \begin{pmatrix} 0{,}48 & 0{,}7 & o & o & o \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 8 \\ 6 \\ 1 \\ 2 \end{pmatrix} - 2 = 7{,}92$$

$$\beta_5 = \begin{pmatrix} 0{,}48 & 0{,}7 & o & o & o \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 6 \\ 8 \\ 1{,}5 \\ 1{,}25 \end{pmatrix} -1{,}5 = 5{,}1$$

This is equal to the multiplication of:

$$\begin{pmatrix} 0{,}48 & 0{,}7 & o & o & o \end{pmatrix} \cdot \begin{pmatrix} 10 & 5 & 5 & 9 & 5 \\ 6 & 6 & 8 & 8 & 6 \\ 4{,}5 & 18 & 9 & 6 & 8 \\ 3 & 2 & 1{,}5 & 1 & 1{,}5 \\ 1{,}5 & 3 & 4 & 2 & 1{,}25 \end{pmatrix} = \begin{pmatrix} o & -0{,}4 & o & 7{,}92 & 5{,}1 \end{pmatrix}$$

Since $\beta_2$ is the largest negative $\beta$, $x_2$ must be introduced. $y_{i2}$ must first defined through:

$$y_{12} = B^{-1} \cdot a_{i2}$$

$$y_{12} = \begin{pmatrix} 0,16 & -0,1 & & & \\ -0,12 & 0,2 & & & \\ 0,26 & -1,35 & 1 & & \\ -0,3 & 0 & & 1 & \\ 0,24 & -0,65 & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 6 \\ 18 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0,2 \\ 0,6 \\ 11,7 \\ 0,5 \\ 0,3 \end{pmatrix}$$

The min nonnegative $p'^{o}_{i} / y_{12}$ is defined

$$p'^{o}_{g} / y_{g2} = p'^{o}_{3} / y_{32}$$

Pivoting on $y_{32}$

| | | | | | $\bar{x}_2$ | $p'^{o}$ |
|---|---|---|---|---|---|---|
| $x_1$ | -0,16 | -0,1 | | | 0,2 | 4,4 |
| $x_3$ | -0,12 | 0,2 | | | 0,6 | 4,2 |
| $b_3$ | -0,36 | -1,35 | 1 | | 11,7 | 23,4 |
| $b_4$ | -0,3 | 0 | | 1 | 0,5 | 10,5 |
| $b_5$ | -0,24 | -0,65 | | | 1 0,3 | 5,6 |
| | -,48 | 0,7 | 0 | 0 | 0 -,4 | 73,2 |

The inverse at the third iteration would be

| | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | $p^o$ |
|---|---|---|---|---|---|---|
| $x_1$ | 0,16 | 0,17 | -0,01 (§) | | | 4 |
| $x_3$ | -0,13 | 0,26 | -0,05 | | | 3 |
| $x_2$ | 0,02 | -0,11 | 0,08 | | | 2 |
| $b_4$ | -0,31 | 0,05 | -0,04 | 1 | | 9,5 |
| $b_5$ | 0,24 | -0,62 | -0,02 | | 1 | 5 |
| | 0,492 | 0,653 | 0,034 | 0 | 0 | 74 |

To examine whether our solution is optimal, we compute $\beta_4$, $\beta_5$ as follows:

$$\beta_4 = \left(\beta_1\ \beta_2\ \beta_3\ \beta_4\ \beta_5\right) \cdot \begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \\ a_{54} \end{pmatrix} - c_4$$

$$\beta_4 = \left(0{,}492 \quad 0{,}653 \quad 0{,}034\ 0\ 0\right) \begin{pmatrix} 9 \\ 8 \\ 6 \\ 1 \\ 2 \end{pmatrix} - 2 = 7{,}865$$

$$\beta_5 = \left(0{,}492,\ 053,\ 0340\ o\ \ o\right) \begin{pmatrix} 5 \\ 6 \\ 8 \\ 1{,}5 \\ 1{,}25 \end{pmatrix} - 1{,}5 = 5{,}15$$

since $\beta_j$ $j = 1,2,\ldots,n$ is positive, our solution is optimal. No further iteration are required.

---

(§) The values are rounded.

Using Kairo method

Recalling that $\beta_z$ is

$$(\beta_1 \ \beta_2) = (0,48 \quad 0,7) \qquad z = 1,2$$

$\beta_s$ could be computed as follows

$$\beta_s = [\beta_1 \quad \beta_2] \cdot \begin{pmatrix} a_{1s} \\ a_{2s} \end{pmatrix} - c_s \qquad s = 2,4,5$$

$$\beta_2 = [0,48 \quad 0,7] \begin{pmatrix} 5 \\ 6 \end{pmatrix} - 7 = -0,4$$

$$\beta_4 = [0,48 \quad 0,7] \begin{pmatrix} 9 \\ 8 \end{pmatrix} - 2 = 7,92$$

$$\beta_5 = [0,48 \quad 0,73] \begin{pmatrix} 5 \\ 6 \end{pmatrix} - 1,5 = 5,1$$

Since $\beta_2$ is the largest negative $\beta_s$, $x_2$ must be introduced. The value of $x_2$ in terms of the basis $A_{zq}^{-1}$ is defined as follows:

First we define $y_{z2}$ through :

$$y_{z2} = A_{zq}^{-1} \cdot a_{z2} \qquad z = 1,2$$

$$\begin{pmatrix} y_{12} \\ y_{22} \end{pmatrix} = \begin{pmatrix} 0,16 & -0,1 \\ -0,12 & 0,2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 0,2 \\ 0,6 \end{pmatrix}$$

second we define $y_{d2}$ through

$$y_{d2} = a_{d2} - A_{rr}^{-1} \cdot y_{z2} \qquad \begin{matrix} d = 3,4,5 \\ z = 1,2 \end{matrix}$$

$$\begin{pmatrix} y_{32} \\ y_{42} \\ y_{52} \end{pmatrix} = \begin{pmatrix} 18 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} ,16 & -,1 \\ -,12 & ,2 \end{pmatrix} \cdot \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

Now we find the minimum nonnegative of the ratio

$$p_i^o \; / \; y_{i2}$$

The min. nonneg. of (

( 4,4/ o,2 - 4,2/ o,6 - 23,4/11,7-
10,5/o,5 - 5,6/ o,3 )

is $p'^{o}_{3} = p'^{o}_{3} / y_{32}$

$= 23. 4 / 11,7 = 2$

$$p'^{o}_{i} = p^{o}_{i} - y_{i2} \cdot p'^{o}_{2} \qquad i \neq g$$

$$\begin{pmatrix} p'^{o}_{1} \\ p'^{o}_{2} \\ p'^{o}_{4} \\ p'^{o}_{5} \end{pmatrix} = \begin{pmatrix} 4.4 \\ 4.2 \\ 10,5 \\ 5,6 \end{pmatrix} - \begin{pmatrix} ,2 \\ ,6 \\ 0,5 \\ ,3 \end{pmatrix} \cdot \cdot \qquad \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

The inverse of the new basis is defined through inverting the following system

$$A_{zq} \qquad \begin{matrix} z = 1,2, \text{ and } 3 \\ q = 1,3, \text{ and } 2 \end{matrix}$$

$$A_{zq} = \begin{bmatrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \end{bmatrix} = \begin{bmatrix} 10 & 5 & 5 \\ 6 & 8 & 6 \\ 4,5 & 9 & 18 \end{bmatrix}$$

$$A_{zq}^{-1} = \begin{bmatrix} y_{11} & y_{13} & y_{12} \\ y_{21} & y_{23} & y_{22} \\ y_{31} & y_{32} & y_{32} \end{bmatrix} = \begin{bmatrix} 0,153 & ,0769 & -,01709 \\ -,1384 & ,2692 & -,0512 \\ 0,0307 & -,115 & -,0854 \end{bmatrix}$$

computing $\beta_z$

$$\beta_z = c_q \cdot A_{zq}^{-1} \qquad \begin{matrix} z = 1,2, \text{ and } 3 \\ q = 1,3, \text{ and } 2 \end{matrix}$$

$$[\beta_1 \ \beta_2 \ \beta_3] = [c_1 \ \ c_3 \ \ c_2] \cdot A_{zq}^{-1}$$

$$= (0,492 \quad 0,653 \quad ,034)$$

To examine the optimality of our solution we apply the rule

$$\beta_s = \beta_z \cdot a_{zs} - c_s \qquad \begin{matrix} s = 4,5 \\ z = 1,2,3 \end{matrix}$$

$$\beta_4 = [0{,}492,\ 0{,}653,\ {,}034] \cdot \begin{pmatrix} 9 \\ 8 \\ 6 \end{pmatrix} - 2 = 1{,}856$$

$$\beta_5 = [0{,}492\ 0{,}653\ {,}034] \cdot \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix} - 1{,}5 = 5{,}15$$

Since $\beta_s$ is positive, no further improvement could be achieved.

It might be interesting here to calculate the operations required to solve our example applying the SI, the RSI, and the KM. According to the estimation of Wagner, the required operations per iteration for the SI and the RSI are $2mn$ and $3m^2+mn$ respectively. (1) Thus our example the simplex method would require 160 operations while the RSI would need 304 operations. Calculating the operations when applying the KM, would show that we needed only to perform 105 operations. In the next chapter, we will try to develop a formula which gives the operations required when the KM is used.

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(1) Wagner,H. M. ," A Comparison of the Original and the Revised Simplex Methods", Operations Research, N. 1957, pp. 361-369

## 3.8 THE KAIRO METHOD : EVALUATED

### 3.8.1 How many operations per iteration are required?

In this chapter, we will try to make a rough estimation of the total number of computations required to perform K iterations when the KM is applied to solve a linear programming problem of (m·n) dimensions. Where m is the number of the constraints and n is the number of the real variables. Unfortunately, it is not possible to compute the absolute number of operations per iteration. Applying the KM, the number of operations differs heavily from one iteration to another according to the dimension of the basis $A_{rr}^{-1}$ .

To move towards an adjacent extreme point, there are six steps must be applied: (§)

1) Determining the relative cost coefficients for the nonbasic variables; $\beta_s$
2) Determining the value of the newcoming variable $x_e$ in terms of the current basis, $y_e$
3) Determining the pivot row
4) Determining the current value of the constant vector $P^o$
5) Determining the inverse of the basis $A_{rr}^{-1}$
6) Determining the relative cost coefficients of the real basic variables $\beta_r$.

---

(§) When discussing the number of arithmetic operations involved in any computation, we will only count multiplications and divisions.

Step 1: To determine $\beta_s$, we need:

$$1/2\ K^2n - 1/3\ K^3 + 1/2\ K^2 - 1/2\ K\ n - 1/6\ K \quad \ldots(1)$$

Step 2: To determine $Y_e$, we need:

$$1/2\ K^2m - 1/2\ K\ m \qquad (2)$$

Step 3: To determine the pivot row, we need:

$$K\ m \qquad (3)$$

Step 4: To determine the new value of $P^{\circ}$, we need

$$K\ m \qquad (4)$$

Step 5: Performing the KAIRO ALGORITHM §§ for matrix inversion $m^3$ operations are required to invert a matrix of m order. The total number of iterations required to invert the successive bases are:

$$T = 1 + 2^3 + \ldots K^3$$

$$T = 1/4\ K^4 + 1/2\ K^3 + 1/4\ K^2 \qquad (5)$$

Step 6: To determine the relative cost coefficients $\beta_r$ for the successive K iterations, we need

$$1/3\ K^3 + 1/2\ K^2 + 1/6\ K \qquad (6)$$

Summing up 1 to 6, the total operations required to perform K iterations would be:

$$1/4\ K^2(K^2+2K+5) + 1/2\ K^2(n+m) + 1/2\ K(3m-n) \qquad (7)$$

If $n = 3\ m$, (7) turns to be

$$1/4\ K^2(K^2+2K+5) + 2\ K^2m \qquad (8)$$

---

§ ) See page(19 ) where the derivation of most of the formulas are discussed in details.

§§) The KAIRO ALGORITHM is presented and discussed on pages 96-98

It must be clear that K in (1 to 8) indicates the number of iterations performed where it is assumed that at every iteration a new variable is introduced and, R; the number of the real variables in the basis is equal to K.

Practically, the dimensions of the inverse $A_{rr}^{-1}$ , as previously defined, does not increase at every iteration. It happens frequently that at the Kth iteration, the dimensions of the basis $A_{rr}$ is of very lower dimensions, especially for large K's. This is due to the nature of the wandering on the surface of the convex set. Where a real basic variable may leave the basis and an exhausted constraint may assume once more a non-zero value. (§)

Another estimation could be made for the number of the operations required to perform one iteration as a function of R; the number of the real variables in the basis.

Let us assume that at the Kth iteration there were R real variables in the basis.

Examining the step 1 to 6, previously presented, the number of the operations required to perform the (K+1)st iteration could be computed as follows:

| | | |
|---|---|---|
| Step 1 | $nR - R^2$ | ( 9) |
| Step 2 | $mR$ | (10) |
| Step 3 | $m$ | (11) |
| Step 4 | $m$ | (12) |
| Step 5 | $(R+1)^3 = R^3 + 3R^2 + 3R + 1$ (*) | (13) |
| Step 6 | $(R+1)^2 = R^2 + 2R + 1$ (*) | (14) |

Summing up the operations required we have:

---

§ ) After many trials on problems of different sizes, the number of the real variables during the course of iteration and in the optimal solution was between 1/6 K and 1/4 K

*) Notice that the basis now is of (R+1) dimensions.

$$R^2(R + 3) + R(n + m + 5) + 2m + 2 \qquad (15)$$

Observing the course of iteration when solving a real problem, the pivot element $a_{ge}$ may be:

case 1: an element of $A_{ds}$

case 2: an element of $A_{rs}$

case 3: an element of $A_{dr}$

case 4: an element of $A_{rr}^{-1}$

| | |
|---|---|
| $A_{rr}^{-1}$ (4) | $A_{rs}$ (2) |
| $A_{dr}$ (3) | $A_{ds}$ (1) |

(§§)

It must be clear that the estimation (15) holds only when $a_{ge}$ is an element of $A_{ds}$ - (case 1) .

In cases 2,3 and 4, we do not need to determine the inverse of the new basis, since we can derive it directly making use of the new properties of the matrix inversion (*).

Furthermore, in case 2, the relative cost coefficients could be determined directly applying the usual transformation process. (§)

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*) This point is discussed in details on page (141).

§) Even in cases (1 and 3) such a step could be done if we find $Y_g = - a_g \cdot A_{rr}^{-1}$, this would lead, however, to the same number of operations when multiplying $C_r \cdot A_{rr}^{-1}$ .

§§) We assume that the inversion of $A_{rr}$ has been performed in place (see p.93)

The operations required in case 2 would be

$$nR + m R + R + 2 m \quad (\text{[illegible]} \qquad (16)$$

In case 3, the operations required would be:

$$R(R + n + 2) + 2 m + 1 \qquad (17)$$

In case 4, the operations required would be:

$$R(n + m - 1) + 2 m - R^2 \quad (\text{[illegible]} \qquad (18)$$

Considering the number of operations required by the revised simplex method to perform an iteration

$$3 m^2 + n m \quad (\text{[illegible]}$$

---

( ) Computing 16, we do not need to perform steps 5 and 6. We must add $R^2$ to update the new basis and R to compute $\beta_r$

( ) Computing 18, we do not need to perform steps 2,5, and 6. We must add the operations required to define $y_{(d)e}$ through $y_{de} = - A_{dr} \cdot a_{re}^{-1}$ - and $(R-1)^2$ to update the basis and (R-1) ro define $\beta_r$. Defining $Y_{de}$ will be discussed, once more, in details when presenting the modified KAIRO METHOD, the subject matter of the next chapter.

) Wagner, H. M., "A Comparison of the Original and the Revised Simplex Methods - Operations Research, 5. 1957, pp. 361-369.

Letting n = 3 m, where the revised simplex method is supposed to be better than the standard simplex method, [illegible] to be (1)

$$6\ m^2$$

which indicates that when dealing with one of the cases 2,3, and 4, the KM will be absolutely better than the revised simplex method even when R is equal to m. Dealing with case 1, the revised simplex method appears to be better. This is due to the fact that in case 1 we have to inverse the basis $A_{rr}$. Anyhow, case 1 could be considered as a natural control for rounding-off errors, since the inverse $A_{rr}^{-1}$ is derived directly from the original $A_{rr}$. Reinversing the basis B is a procedure frequently performed when applying the SM or one of its variants to keep the rounding-off errors within bounds. (2)

However, the modified KAIRO METHOD, the subject matter of the next chapter will try to avoid the troublesome job of inverting the basis if the case 1 is to be applied. (§)

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1) Wagner, H. M., op. cit., p. 217

2) Hadley, G. op. cit., p. 216

§) The advantages of the KM will be discussed in details when discussing the Modified KAIRO METHOD.

## 4. THE MODIFIED FORM OF KAIRO METHOD, MKM.

### .1 Introduction:

While the KM is an introduction to the modified form, the latter is a successful method for avoiding its basic disadvantage, namely, the need to invert an "increasing" matrix. Such a process must be very time consuming even if performed on a digital computer.

Recalling step(iiii) page (59) it might become clear that our problem is one of finding the inverse of a matrix enlarged by one column and one row, while the inverse of the old matrix is known.

Instead of inverting the original elements of the enlarged matrix, we have developed a new algorithm to determine the new inverse of the enlarged matrix using the inverse of the original one.

### .2. Matrix inversion:

The inversion of matrices is the same as the division of ordinary numbers, but it is a very much more complicated process. (1)
The inverse of a matrix $A$, or $A^{-1}$, is a matrix which has the property

$$A \cdot A^{-1} = A^{-1}A = I$$

The process which is most widley used, is that of triangulation and back-substitution developed by Gauss and Jordan. When applied to the inversion of a matrix of m order, the method consists of m passes, each having the following three steps. (2)

---

1) Kucy Joan Slater, "Fortran Programs for Economists", The Syndics of the Cambridge University Press, 1967, p. 14

2) SHAN s. KUO, "Numerical Methods and computers", Addison-Wesley, Reading, Massachusetts, 1965, p. 163

1) intercharge of rows

2) normalization

3) elimination

We will now consider the inversion process using the Gauss-Jordan reduction. The intercharge of rows will be ignored to simplify the presentation.

If a matrix $[A]$ is known and its inverse to be found - assuming that it has one - we append a unit matrix of the same order, forming a double square matrix as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix}$$

We start to normalize row 1 and eliminate $a_{21}$ and $a_{31}$. The result of the first iteration is:

$$\begin{bmatrix} 1 & a_{12}/a_{11} & a_{13}/a_{11} & 1/a_{11} & 0 & 0 \\ 0 & a_{22}-a_{21}(a_{12}/a_{11}) & a_{23}-a_{21}(a_{13}/a_{11}) & -a_{21}/a_{11} & 1 & 0 \\ 0 & a_{32}-a_{31}(a_{12}/a_{11}) & a_{33}-a_{31}(a_{13}/a_{11}) & -a_{31}/a_{11} & 0 & 1 \end{bmatrix}$$

By continuing the process to eliminate the other variables, we obtain the following form:

$$\begin{bmatrix} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{bmatrix}$$

In this resultant matrix, the matrix $[B]$ is the inverse of $[A]$.

The process explained above has been modified to save a portion of computer storage. In the modified form the elements of $A^{-1}$ are to be stored, as computed, in a location in A ; and hence the appended matrix is not needed. This new modification is frequently known as matrix inversion in place.(1)

Numerical example:

$$\text{Let } A = \begin{pmatrix} 3 & -6 & 7 \\ 9 & 0 & -5 \\ 5 & -8 & 6 \end{pmatrix}$$

The following tableau shows a step-by-step how the inversion process is accomplished.

(1) SHAN S. KUO, op cit , p. 164

Table 8.5 Matrix inversion in place(§)

| Column 1 [A] reproduced from Table 8.4 | | | Column 2 Inversion in place | | |
|---|---|---|---|---|---|
| Pass 1, Step 1 | | | | | |
| [1] | −2 | 2.33 | [0.33] | −2 | 2.33 |
| 9 | 0 | −5 | 9 | 0 | −5 |
| 5 | −8 | 6 | 5 | −8 | 6 |
| Step 2 | | | | | |
| 1 | −2 | 2.33 | 0.33 | −2 | 2.33 |
| [0] | 18 | −26 | [−3] | 18 | −26 |
| 5 | −8 | 6 | 5 | −8 | 6 |
| Step 3 | | | | | |
| 1 | −2 | 2.33 | 0.33 | −2 | 2.33 |
| 0 | 18 | −26 | −3 | 18 | −26 |
| [0] | 2 | −5.65 | [−1.65] | 2 | −5.65 |
| Pass 2, Step 1 | | | | | |
| 1 | −2 | 2.33 | 0.33 | −2 | 2.33 |
| 0 | [1] | −1.44 | −0.17 | [0.06] | 1.44 |
| 0 | 2 | −5.65 | −1.65 | 2 | −5.65 |
| Step 2 | | | | | |
| 1 | [0] | −0.55 | −0.01 | [0.12] | 0.55 |
| 0 | 1 | −1.44 | −0.17 | 0.06 | 1.44 |
| 0 | 2 | −5.65 | −1.65 | 2 | −5.65 |
| Step 3 | | | | | |
| 1 | 0 | −0.55 | −0.01 | 0.12 | 0.55 |
| 0 | 1 | −1.44 | −0.17 | 0.06 | 1.44 |
| 0 | [0] | 2.77 | 1.31 | [−0.12] | 2.77 |
| Pass 3, Step 1 | | | | | |
| 1 | 0 | −0.55 | −0.01 | 0.12 | −0.55 |
| 0 | 1 | −1.44 | −0.17 | 0.06 | 1.44 |
| 0 | 0 | [1] | 0.48 | [0.04] | 0.36 |
| Step 2 | | | | | |
| 1 | 0 | −0.55 | −0.01 | 0.12 | −0.55 |
| 0 | 1 | [0] | 0.52 | 0.12 | [−0.52] |
| 0 | 0 | 1 | 0.48 | 0.04 | −0.36 |
| Step 3 | | | | | |
| 1 | 0 | [0] | 0.26 | 0.14 | [0.20] |
| 0 | 1 | 0 | 0.52 | 0.12 | −0.52 |
| 0 | 0 | 1 | 0.48 | 0.04 | −0.36 |

(§) The example has been taken from SUAN S. KUO, op cit, p. 165

The inversion of any matrix, as we have seen, is a complicated one, and the number of steps involved increases very quickly as the size of the matrix increases.
Thus we need about

$$4 \cdot 3^3 + 4 \cdot 2^3 = 140 \text{ operations}$$

to invert a 3 x 3 matrix, but we need about

$$4 \cdot 4^3 + 4 \cdot 3^3 + 4 \cdot 2^3 = 396 \text{ operations}$$

to invert a 4 x 4 matrix.

In general we need to carry out about

$$4(1^3 + 2^3 + \ldots + n^3) = 4 \sum_{v=1}^{m} v^3 = m^2(m+1)^2$$

operations in order to invert an m x m matrix . (1)

There are also many other methods for computing an inverse of a matrix. They are, however, appropriate only for small matrices . (2)

Examining the Gauss-Jordan elimination process, it would be clear that such a procedure does not enable us to derter-mine the inverse of $A'^{-1}$ when $A^{-1}$ is available without sol-ving the original matrix A from the beginning where A' is the same as A but enlarged with an extra column vector and row vector.

$$A = \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$$A' = \begin{matrix} & A & & a_{14} \\ & & & a_{24} \\ & & & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{matrix}$$

---

1) Lucy Joan Slater, op.cit. pp. 19-20
2) See R.V. Andree, "Computation of the Inverse of a matrix", American Math. monthly, Volume LVIII(1951), pp. 87 - 92 and matrix Inversion by the use of determinant ( di Rocca ferrera, op.cit. pp. 170 - 172)

The KAIRO ALGORITHM, the subject matter of the next discussion, can, however, invert an enlarged matrix without reconsidering the original matrix from the beginning. Furthermore, it requires fewer operations than required by the Gauss-Jordan procedure.

## 4.3 Kairo Algorithm for matrix inversion: (§)

Let

$$A_{rr} = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \\ a_{i1} & \cdots & a_{ik} \\ \vdots & & \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}$$

and

$$A_{rr}^{-1} = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \\ a_{i1} & \cdots & a_{ik} \\ \vdots & & \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}^{-1}$$

---

(§) The reader will find a similar procedure in (Zielke , G., " Numerische Berechnung von benachbarten inversen Matrizen und linearen Gleichungssystemen, Friedr. Vieweg& Sohn. Braunschweig, 1970,pp 33-35). Indeed, we have developed Kairo Algorithm long before we read it by Zielke. However, the procedure as presented in Zielke's book, is nothing more than a special case. Kairo Algorithm is, on the contrary, a complete procedure to inverse square nonsingular matrices. Moreover, the Kairo Algorithm has been modified to determine the value of the variable to be introduced into the basis where only the inverse of the submatrix $A_{rr}$ is available.

Enlarging the original matrix $A_{rr}$ with one more column and row as follows:

$$A_{ee} = \begin{vmatrix} a_{11} & & a_{1k} & a_{1,k+1} \\ \cdots & \cdots & \cdots & \cdot \\ a_{ik} & \cdots & a_{ik} & a_{i,k+1} \\ \cdots & \cdots & \cdot & \cdot \\ a_{k1} & \cdots & a_{kk} & a_{k,k+1} \\ a_{k+1,1} & \cdots & a_{k+1,k} & a_{k+1,k+1} \end{vmatrix}$$

The inverse of $A_{ee}$ could be obtained applying the following steps:

1. Multiply the new column $a_{r,k+1}$ $\quad r = 1,2,\ldots,k$

   with the inverse of $A_{rr}$ , the product is

$$\bar{a}_{r(k+1)} = A_{rr}^{-1} \cdot a_{r(k+1)}$$

2. Multiply $\bar{a}_{r,k+1}$ with the first k elements of the annexed row $a_{k+1,r}$

$$\begin{matrix} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{vmatrix} & \begin{matrix} \bar{a}_{1,k+1} \\ \bar{a}_{2,k+1} \\ \\ \bar{a}_{k,k+1} \end{matrix} \\ a_{k+1,1} \quad a_{k+1,2} \quad \cdots \quad a_{k+1,k} & ? \end{matrix}$$

$$S = a_{k+1,r} \cdot \bar{a}_{r',k+1}$$

3. Define the new pivot element through

$$\bar{a}_{(k+1),(k+1)} = a_{(k+1),(k+1)} - S$$

4. Multiply $a_{(k+1),r}$ by $A_{rr}^{-1}$, the negative of the product is $\bar{a}_{k+1,r}$

$$\bar{a}_{(k+1),r} = - a_{(k+1),r} \cdot A_{rr}^{-1}$$

5. Finally, perform the usual transformation process as follows:

$$u = k + 1$$

a) $y_{ri} = \bar{a}_{ri} - \bar{a}_{ru} \cdot \bar{a}_{ui} / \bar{a}_{uu}$

$r=1,2,\ldots,k \qquad i=1,2,\ldots,k$

b) $y_{ur} = \bar{a}_{ur} / \bar{a}_{uu}$

$r=1,2,\ldots,k$

c) $y_{ru} = - \bar{a}_{ru} / \bar{a}_{uu}$

$r=1,2,\ldots,k$

d) $y_{uu} = 1. / \bar{a}_{uu}$

Numerical example:

Find the inverse of this matrix

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

Step 1:

Starting with $a_{11}$ its inverse is

$$\bar{a}_{11} = 1/\ a_{11}$$
$$0,5 = 1/\ 2$$

Step 2:

Multiplying $\bar{a}_{11}$ with $a_{12}$

$$\bar{a}_{12} = \bar{a}_{11} \cdot a_{12}$$
$$0,5 = 0,5 \ . \ 1$$

Step 3:

Multiply $\bar{a}_{12}$ with $a_{21}$ the product is S

$$S = \bar{a}_{12} \cdot a_{21}$$
$$1,5 = 0,5 \ . \ 3$$

Step 4:

Define the value of the new pivot through:

$$\bar{a}_{22} = a_{22} - S$$
$$0,5 = 2 - 1,5$$

Step 5:

$$\bar{a}_{21} = -a_{21} \cdot \bar{a}_{11}$$
$$-1,5 = -3 \cdot 0,5$$

Step 6:

Pivoting on $\bar{a}_{22}$ the inverse of A is

$$\begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{pmatrix} = \begin{pmatrix} 0,5 & 0,5 \\ -1,5 & 0,5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

A SPECIAL CASE OF MATRIX INVERSION. (1)

To determine the inversion of a matrix of (2x2) dimensions

1. define the determinant D

$$D = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

2. the inverse $A^{-1}$ is

$$A^{-1} = \begin{pmatrix} a_{22}/D & -a_{12}/D \\ -a_{21}/D & a_{11}/D \end{pmatrix}$$

(1) Hadley, G., op cit p. 34

Using the same example

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

$$D = 2.2 - 3.1$$

$$A^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Let us now assume that original matrix A is enlarged as follows:

$$\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 4 \\ 1 & 3 & 2 \end{pmatrix}$$

Using Kairo Algorithm and replacing the inverse of $A^{-1}$ instead of A.

$$\begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & 4 \\ 1 & 3 & 2 \end{pmatrix}$$

applying steps 1 to 5(pp.97-98) the matrix will be as follows:

$$\begin{pmatrix} 2 & -1 & -4 \\ -3 & 2 & 8 \\ 7 & -5 & -18 \end{pmatrix}$$

applying the usual transformation process pivoting on $\bar{a}_{33}$ , in our case -18, the inverse of $A_{ee}$ is

$$A_{ee}^{-1} = \begin{pmatrix} 4/9 & 1/9 & -2/9 \\ 1/9 & -2/9 & 4/9 \\ -7/18 & 5/18 & -1/18 \end{pmatrix}$$

Proof:

$$I = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 4 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 4/9 & 1/9 & -2/9 \\ 1/9 & -2/9 & 4/9 \\ -7/18 & 5/18 & -1/18 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The new algorithm, being a cyclic, i.e., a repetitive process can be programmed fairly easily.

A computer code for matrix inversion based on FORTRAN IV has been written by the author to execute the new algorithm as previously explained. The code is based on the following steps: (§)

(§) Writing this code and the next one, our interest was not devoted to write optimal codes from the programming point of view.

```
```

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PAGE   2

END OF COMPILATI[illegible]

// XE[illegible]



  THE [illegible]AT[illegible]IX [illegible]

  0.44444450, [illegible] [illegible].1111111[illegible]5E [illegible] -[illegible].[illegible]

  0.[illegible]945[illegible] - .[illegible]  .[illegible]

-[illegible].[illegible]  [illegible].[illegible]  - .[illegible]
```

This code has the following disadvantages:

1. The first element of the matrix must be of non-zero value
2. the programm does not advise the user if his matrix is singular.

It has been suggested, that pivoting on the largest absolute element in the matrix at each cycle (1) performs as a guard against the increasing loss of significant figures in our calculation and against the implicit division by a near-zero quantity. (2) Applying the KAIRO algorithm, our ability to choose the largest absolute element in the matrix is very limited. Simply because only the first $k^2$ elements would have been treated at the k iteration; the other elements $a_{ij}$

(i or j or both of them greater than k)

are still untouched. When applying the Gauss and Jordan procedure all $a_{ij}$ $i = 1,2,...,n$ $j=1,2,...,m$ are subject to the elimination process at every pass.
Fortunately a compromising process could be adapted in order to:

1. avoid division by small elements.
2. to pivot on the largest absolute element in the column $y_{ik}$ where $i = 1,2,...,m$ and the index k implies the number of the iteration performed.

A more sophistivated code has been written by the writer. The new code performs a kind of pivot-row-selection to avoid pivoting on the near-zero values. Furthermore, the new code advises the user if his matrix is singular.

---

1) Lucy Joan Slater, "Fortran Programms for Economics", Cambridge University Press, 1967, p. 19
2) Ibid p-19

```
PAGE    1

// JOB

LOG DRIVE   CART SPEC   CART AVAIL  PHY DRIVE
      1         0007        0007        0003

V2 M19   ACTUAL  8K  CONFIG  8K

// FOR
*LIST SOURCE PROGRAM
*IOCS (1132 PRINTER,CARD, ISK)
*ONE WORD INTEGERS
*NAME MAIN
C       THE MODIFIED KAIRO ALGORITHM FOR MATRIX INVERSION
C                    BY MUSTAFA Y. GOUDA
        DIMENSION A(20,20),COF(20),INDEX(20),AY(20,20),E(20,20)
        READ(2,100) M
        DO 30 I=1,M
30      READ(2,200) (A(I,J),J=1,M)
        DO 133 I=1,M
        DO 133 J=1,M
133     AY(I,J)=A(I,J)
        WRITE (3,300)
        DO 40 I=1,M
40      WRITE(3,400) (A(I,J),J=1,M)
C       STATEMENTS FOR SEARCH THE LARGEST ABSOLUTE  ELEMENT IN THE MATRI
        K=1
        PIVOT=0.
        DO 3 I=1,M
        DO 3 J=1,M
        IF(ABS(PIVOT) - ABS(A(I,J))) 2,3,3
2       PIVOT=A(I,J)
        IROW=I
        ICOL=J
3       CONTINUE
C       STATEMENTS TO PUT THE LARGEST ABSOLUTE ELEMENT ON A(1,1) POSITIO
        DO 54 I=1,M
        T=A(I,ICOL)
        A(I,ICOL)=A(I,K)
54      A(I,K)=T
        IF(IROW-K) 4,6,4
4       DO 5 L=1,M
        T=A(IROW,L)
        A(IROW,L)=A(1,L)
5       A(1,L)=T
6       INDEX(K)=IROW
C       INVERTING THE A(1,1)
        A(K,K)= 1./A(K,K)
        DO21 K=2,M
        KE=K-1
        DO 8 J=1,KE
        COF(J)=0.
        DO 8 I=1,KE
8       COF(J)=COF(J)+A(I,K)*A(J,I)
        DO 10 I=K,M
        COF(I)=0.
        DO 9 J=1,KE
9       COF(I)=COF(I)+COF(J)*A(I,J)
10      COF(I)=A(I,K)-COF(I)
C       STATEMENT FOR SEARCH THE LARGEST ABSOLUTE ELEMENT IN THE NEW COL
C       COLUMN TO USE IT AS A NEXT PIVOT
        PIVOT=0.
        DO 12 I=K,M
```

```
      IF(ABS(PIVOT) - ABS(COF(I))) 11,12,12
11    IROW=I
      P=COF(I)
12    CONTINUE
      IF(ABS(P)-1.E-7) 701,701,700
C     STATEMENTS TO INTERCHANGE ROWS- TO PUT THE PIVOT ON DIAGONAL
700   IF(IROW-K) 13,15,13
13    DO 14 J=1,M
      T=A(IROW,J)
      A(IROW,J)=A(K,J)
14    A(K,J)=T
15    INDEX(K)=IROW
      A(K,K)=1./P
C     STATEMENTS TO PERFORM THE TRANSFORMATION PROCESS
      DO 16 I=1,KE
16    A(I,K)= - COF(I)*A(K,K)
      DO 17 J=1,KE
      COF(J)=0.
      DO 17 I=1,KE
17    COF(J)=COF(J)-A(K,I)*A(I,J)
      DO 19 L=1,KE
      DO 19 I=1,KE
19    A(L,I)=A(L,I)+A(L,K)*COF(I)
      DO 20 I=1,KE
20    A(K,I)=COF(I)*A(K,K)
21    CONTINUE
C     STATEMENTS TO INTERCHANGE COLUMNS
      DO 31 JJ=1,M
      J=M-JJ+1
      IF(INDEX(J)-J) 33,31,33
33    IROW=INDEX(J)
      IR=J
      DO 34 J=1,M
      T=A(J,IR)
      A(J,IR)=A(J,IROW)
      A(J,IROW)=T
34    CONTINUE
31    CONTINUE
      DO 55 J=1,M
      T=A(ICOL,J)
      A(ICOL,J)=A(1,J)
55    A(1,J)=T
      WRITE (3,500)
      DO 50 I=1,M
50    WRITE(3,400) (A(I,J),J=1,M)
C     STATEMENTS TO DEFINE THE IDENTITY MATRIX
      DO 45 I=1,M
      DO 45 J=1,M
      E(I,J)= 0.
      DO 45 K=1,M
45    E(I,J)=E(I,J)+A(I,K)*AM(K,J)
      WRITE (3,405)
405   FORMAT('1FINDHEITSMATRIX')
      DO 41 I=1,M
41    WRITE(3,130) (E(I,J),J=1,M)
      GO TO 999
701   WRITE(3,1006)
100   FORMAT(I3)
130   FORMAT (//,1HO(5E15.8))
200   FORMAT(20F4.1)
300   FORMAT('1 THE ORIGINAL MATRIX')
400   FORMAT(//,1HO(4F15.8))
```

```
      FORMAT('1 THE MATRIX INVERSION USING KAIRO ALGORITHM')
 1005 FORMAT('1 THE MATRIX IS SINGULAR')
 999  CALL EXIT
      END
```

END OF COMPILATION

// XEQ

THE ORIGINAL MATRIX

0.20000004E 01 0.10000002E 01 0.00000000E 00

0.30000004E 01 0.20000004E 01 0.40000009E 01

0.10000002E 01 0.30000004E 01 0.20000004E 01

THE MATRIX INVERSION USING KAIRO ALGORITHM

0.44444447E 00 0.11111111E 00-0.22222223E 00

0.11111111E 00-0.22222223E 00 0.44444447E 00

-0.38888889E 00 0.27777785E 00-0.55555589E-01

EIGENFITS MATRIX

## 4.4 Proof of the KAIRO ALGORITHM

Let $\sum_{r=1}^{k} B_{rr}^{-1}$ be the inverse of $B_{rr}$

where

$$B_{rr} = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1k} \\ a_{21} & a_{22} \cdots & a_{2k} \\ \vdots & \vdots & \\ a_{k1} & a_{k2} \cdots & a_{kk} \end{pmatrix}$$

and

$$B_{rr}^{-1} = \begin{pmatrix} y_{11} & y_{12} \cdots & y_{1k} \\ y_{21} & y_{22} \cdots & y_{2k} \\ \vdots & & \\ y_{k1} & y_{k2} \cdots & y_{kk} \end{pmatrix}$$

now let $a_{dr}$ be an additional row vector added to $B_r$. The new basis for the enlarged problem $B_d$ would contain k+1 vectors. Then

$$B_d = \begin{pmatrix} B_{rr} & 0 \\ a_{dr} & \pm 1 \end{pmatrix}$$

and

$$B_d^{-1} = \begin{pmatrix} B_{rr}^{-1} & 0 \\ -a_{dr}B_r^{-1} & 1 \end{pmatrix}$$

Now let $\hat{B}$ be another basis where

$$\hat{B} = \begin{pmatrix} B_{rr} & a_{rd} \\ a_{dr} & a_{dd} \end{pmatrix}$$

and $\begin{pmatrix} a_{rd} \\ a_{dd} \end{pmatrix}$ is a column vector contains k+1 elements.

Let us assume that

$$B_d^{-1} \cdot \hat{B} = E$$

$$\begin{pmatrix} B_{rr}^{-1} & 0 \\ -a_{dr}B_{rr}^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} B_{rr} & a_{rd} \\ a_{dr} & a_{dd} \end{pmatrix} = E$$

where

$$E = \begin{pmatrix} I & B_{rr}^{-1}a_{rd} \\ 0 & a_{dd} - a_{dr}B_{rr}^{-1}a_{rd} \end{pmatrix}$$

and

$$E^{-1} = \begin{pmatrix} I & \dfrac{-B_{rr}^{-1}\, a_{rd}}{a_{dd} - a_{dr}B_{rr}^{-1}\, a_{rd}} \\ 0 & \dfrac{1}{a_{dd} - a_{dr}B_{rr}^{-1}a_{rd}} \end{pmatrix}$$

Multiplying the equation

$$B_d^{-1}\, \hat{B} = E \quad \text{by} \quad \hat{B}^{-1}$$

we have

$$B_d^{-1} = \hat{B}^{-1}\, E$$

and

$$\hat{B}^{-1} = E^{-1}\, B_d^{-1}$$

then

$$\hat{B}^{-1} = \begin{pmatrix} 1 & \dfrac{-B_{rr}^{-1}a_{rd}}{a_{dd}-a_{dr}B_{rr}^{-1}a_{rd}} \\ 0 & \dfrac{1}{a_{dd}-a_{dr}B_{r}^{-1}a_{rd}} \end{pmatrix} \cdot \begin{pmatrix} B_{rr}^{-1} & 0 \\ -a_{dr}B_{rr}^{-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} B_{rr}^{-1} + \dfrac{B_{rr}^{-1}a_{rd}a_{rd}B_{rr}^{-1}}{a_{dd}-a_{dr}B_{rr}^{-1}a_{rd}} & \dfrac{-B_{rr}^{-1}a_{rs}}{a_{dd}-a_{dr}B_{rr}^{-1}a_{rd}} \\ \dfrac{-a_{dr}B_{rr}^{-1}}{a_{dd}-a_{dr}B_{rr}^{-1}a_{rd}} & \dfrac{1}{a_{dd}-a_{dr}B_{r}^{-1}a_{rd}} \end{pmatrix}$$

Performing the KAIRO ALGORITHM we would have reached the same result through:

(1) multiplying $\begin{pmatrix} a_{rd} \\ a_{dd} \end{pmatrix}$, the new column vector, with $B_d^{-1}$

$$\begin{pmatrix} B_{rr}^{-1} & 0 \\ -a_{dr}B_{rr}^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{rd} \\ a_{dd} \end{pmatrix} = \begin{pmatrix} B_{rr}^{-1}a_{rd} \\ a_{dd}-a_{dr}B_{rr}^{-1}a_{rd} \end{pmatrix}$$

(2) Pivoting on $a_{dd} - a_{dr}B_{rr}^{-1}a_{rd}$

$$\begin{pmatrix} B_{rr}^{-1} & 0 \\ -a_{dr}B_{rr}^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} B_{rr}^{-1}a_{rd} \\ a_{dd}-a_{dr}B_{rr}^{-1}a_{rd} \end{pmatrix}$$

we obtain $\hat{B}^{-1}$ where

$$\hat{B}^{-1} = \begin{pmatrix} B_{rr}^{-1} + \dfrac{B_{rr}^{-1} a_{rd} a_{rd} B_{rr}^{-1}}{a_{dd} - a_{dr} B_{rr}^{-1} a_{rd}} & \dfrac{-B_{rr}^{-1} a_{rd}}{a_{dd} - a_{dr} B_{rr}^{-1} a_{rd}} \\ \dfrac{-a_{dr} B_{rr}^{-1}}{a_{dd} - a_{dr} B_{rr}^{-1} a_{rd}} & \dfrac{1}{a_{dd} - a_{dr} B_{rr}^{-1} a_{rd}} \end{pmatrix}$$

now let $D = a_{dd} - a_{dr} B_{rr}^{-1} a_{rd}$

then

$$\hat{B}^{-1} = \begin{pmatrix} B_{rr}^{-1} + \dfrac{B_{rr}^{-1} a_{rd} a_{rd} B_{rr}^{-1}}{D} & \dfrac{-B_{rr}^{-1} a_{rd}}{D} \\ \dfrac{-a_{dr} B_{rr}^{-1}}{D} & \dfrac{1}{D} \end{pmatrix}$$

The question now is : is it possible to define $B_{rr}^{-1}$ if $\hat{B}^{-1}$ is known - where $\hat{B}$ -as assumed- is equal to $B_{rr}$ plus an additional column and an additional row? (§)

It is interesting to indicate that pivoting on the element $\frac{1}{D}$, or in other words; choosing the additional column as a pivot column and the last row as a pivot row, we obtain

$$\begin{pmatrix} B_{rr}^{-1} & + B_{rr}^{-1} a_{rd} \\ -a_{dr} B_{rr}^{-1} & D \end{pmatrix}$$

(§) In the available literature, this question has never been answered.

Such a conclusion means that having the inverse $B^{-1}$ where

$$B = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1k} \\ a_{21} & a_{22} \cdots a_{2k} \\ \vdots & \\ a_{k1} & a_{k2} \cdots a_{kk} \end{pmatrix}$$

and

$$B^{-1} = \begin{pmatrix} y_{11} & y_{12} \cdots y_{ik} \\ y_{21} & y_{22} \cdots y_{2k} \\ \vdots & \\ y_{k1} & y_{k2} \cdots y_{kk} \end{pmatrix}$$

it is possible:

(1) To define the inverse $\hat{B}^{-1}$ where $\hat{B}$ is equal to B plus an additional column and row

$$\hat{B} = \left(\begin{array}{ccc|c} a_{11} & a_{12} \cdots a_{1k} & & a_{1,k+1} \\ a_{21} & a_{22} \cdots a_{2k} & & a_{2,k+1} \\ \vdots & & & \\ a_{k1} & a_{k2} \cdots a_{kk} & & a_{k,k+1} \\ \hline a_{k+1,1} & a_{k+1,2} \cdots & a_{k+1,k} & a_{k+1,k+1} \end{array}\right)$$

(2) Having defined $\hat{B}^{-1}$

$$\hat{B}^{-1} = \left(\begin{array}{ccc|c} y_{11} & y_{12} \cdots y_{1k} & & y_{1,k+1} \\ y_{21} & y_{22} \cdots y_{2k} & & y_{2,k+1} \\ \vdots & & & \\ y_{k1} & y_{k2} \cdots y_{kk} & & y_{k,k+1} \\ \hline y_{k+1,1} & y_{k+1,2} & y_{k+1,k} & y_{k+1,k+1} \end{array}\right)$$

to define $B^{-1}$ once more by pivoting on the element $y_{k+1,k+1}$

(3) If we consider the effect of the interchanging of rows and columns, our conclusion could be generalized so that having an inverse $\hat{B}^{-1}$, and pivoting on the element $y_{ij}$ and neglecting both of the i+h row and j+h column, we obtain the inverse $B^{-1}$ where B is equal to $\hat{B}$ minus the j+h row and the i+h column.

Matrix Structure

"The matrices encountered in practical problems usually exhibit some kind of structure and also exhibit another phenomenon of interest - they are very sparcely populated with nonzero elements".(1) A matrix of the form

$$M = \begin{pmatrix} B_1 & 0 & 0 \ldots & 0 \\ 0 & B_2 & 0 \ldots & 0 \\ & & & \vdots \\ 0 & 0 & B_3 & \\ 0 & 0 & 0 \ldots & B_n \\ A_1 & A_2 & A_3 \ldots & A_n \end{pmatrix}$$

"is increasingly more prevalent because of decentralization of decision making in substantial numbers of organizations of all types. We also get this type of structure as a result of a discrete model consisting of quasi-independent linear programs over several time periods linked by one or more constraints." (2)

---

(1) Garvin, W., op cit. p. 81

(2) Hans G. Daellenbach & Earl Bell, "User's Guide to LP", Prentice-Hall, Inc. 1970, p. 151

Grochla, G., Unternehmungsorganisation, Reinbek b Hamburg, Rowohlt 1972.

Hax, H., Die Koordination von Entscheidungen, Köln,

Berlin. 1965.

Lücke, W., Produktion- und Kostentheorie, Physica Verlag, Würzburg-Wien, 1973

Inverting such a matrix with the usual methods may be "equivalent to shooting a rabbit with an elephant gun". Garvin added that we are forced to do that until special guns are designed.[1]. We are going to present here a "special gun" which enables us to invert such structured matrices effectivelly. Moreover, the "new procedure" can be employed to invert the very large scale matrices on relative small computers.

Assuming that we have a matrix of the form

$$M = \begin{pmatrix} B_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & B_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & B_3 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & B_n & 0 \\ A_1 & A_2 & A_3 & \cdots & A_n & 1 \end{pmatrix}$$

where $B_j$, $j=1,2,\ldots,n$ square nonsingular matrices, the inverse of such a matrix would be:(§)

$$M^{-1} = \begin{pmatrix} B_1^{-1} & 0 & 0 & 0 & 0 \\ 0 & B_2^{-1} & 0 & 0 & 0 \\ 0 & 0 & B_3^{-1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & B_n^{-1} & 0 \\ -A_1B_1^{-1} & -A_2B_2^{-1} & -A_3B_3^{-1} & -A_nB_n^{-1} & 1 \end{pmatrix}$$

Proof:

$$M \cdot M^{-1} = I$$

---

1) Garvin, W.W., op.cit p. 82

§) In the available literature, there is no such an explicit inversion.

Hence, to invert such a matrix, it is required only to invert the submatrices $B_1$, $B_2$,...,$B_n$ and to define $-A_1 \cdot B_1^{-1}$, $-A_2B_2^{-1}$,...,$-A_nB_n^{-1}$. Provided that each of $B_j$ $j = 1,2,...,n$ are square matrices and have an inverse.

Such a fact would enable us to invert matrices of the form M having high dimensions on relative small computers.

## 4.6 How many operations are required to ...

Let us now calculate the number of operations required to invert a matrix of m order.

1. Operations required to transform the increasing system $A_{kk}$ , where k implies the number of the iterations performed.

   - at the first iteration one operation is required
   - at the second iteration operations are required
   - at the k+h iteration $k^2$ operations are required
   - at the m iteration $m^2$ operations are required.

Such a relation is governed by the law:

$$\text{Total} = {}^1/_6\ m\ (m+1)(2m+1)$$

where m is the order of the matrix

$$\text{Total} = {}^1/_3\ k^3 + {}^1/_2\ k^2 + {}^1/_6\ k \qquad (1)$$

2. Operations required to define the new subvector column $a_{rk}$ where $r = 1,2,\ldots,k$ and k implies the index of the new column vector.

   - at the first iteration one operation is required
   - at the second iteration 4 operations are required
   - at the k+h iteration $k^2$ operations are required
   - at the (m-1) iteration $(m-1)^2$ operations are required
   - at the m iteration 10 operations are required

$$\text{Total} = {}^1/_6\ (m-1)(m)(2(m-1)+1)$$

$$= {}^1/_3\ k^3 - {}^1/_2\ k^2 + {}^1/_6\ k \ldots \qquad (2)$$

3. Operations required to define the new subvector row

$a_{kr}$

where $r = 1,2,\ldots,k\,1$

and k is the index of the row-vector and the number of iterations performed.

- at the first iteration one operation is required
- at the second iteration 4 operations are required
- at the k+h iteration $k^2$ operations are required
- at the (m-1) iteration $(m-1)^2$ operations are required
- at the m iteration n operations are required

The Total is

$$= {}^1/_2\, k^3 - {}^1/_2\, k^2 + {}^1/_6\, k \ldots \qquad (3)$$

4. Operations required to define the pivot element $a_{kk}$.

- at the first iteration 1 operation is required
- at the second iteration 2 operations are required
- at the k+h iteration k operations are required
- at the (m-1) iteration (m-1) operations are required
- at the m iteration no operations are required

The Total is

$$= {}^1/_2\, k^2 - {}^1/_2\, k \qquad \ldots \qquad (4)$$

Summing up 1 to 4, the total operations required to inverse a matrix of m order is

$$m^3$$

compared with the operations required to obtain the inverse of m order when the Gauss-Jordan procedure is used, it might be clear how superior the new algorithm is.(§)

## 47 MKM : its theoretical [illegible].

We will assume that an optimal solution has been already found for the follwing LP problem:

$$\max. \quad CX$$

subject to
$$AX \leq b^O$$
and
$$X \geq O$$

The associated basis for the solution is:

$$B = P_1 , P_2, \ldots, P_k = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ a_{i1} & \cdots & a_{ik} \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}$$

and

$$B^{-1} = \bar{P}_1 , \bar{P}_2, \ldots, \bar{P}_k = \begin{pmatrix} y_{11} & \cdots & y_{1k} \\ y_{i1} & \cdots & y_{ik} \\ y_{k1} & \cdots & y_{kk} \end{pmatrix}$$

The value of the optimal solution is:

$$\bar{X} = \bar{P}^O = \begin{pmatrix} P_1^O \\ \vdots \\ P_1^O \\ \vdots \\ P_k^O \end{pmatrix}$$

(§) A modification of the Gauss-Jordan procedure requires $m^3$ , see Müller-Merbach, Operations Research, op cit pp. 223-225

and the simplex multipliers are:

$$\beta = \beta_1 , \beta_2, \ldots, \beta_k$$

Now let us assume further, that the constraint

$$a_{g1} + a_{g2} + \ldots + a_{gi} + \ldots + a_{gk} \leq b_g$$

where $g = k + 1$

has to be added to the system. Such a case is already being treated as one of the post-optimality problems, when a new constraint has to be considered. (1) The addition of such a constraint will either lead to decrease in the objective function or leave it unchanged. (2) Let the optimal solution satisfy the new constraint. The immediate effect of this addition is to increase the dimension of the basis from k to k+1. Call the new enlarged basis $\bar{B}$ and the new constraint U, then,

$$\bar{B} = \begin{vmatrix} a_{11} & \ldots & a_{1k} & 0 \\ \ldots & \ldots & \ldots & \cdot \\ a_{k,1} & & a_{kk} & 0 \\ a_{g1} & & a_{gk} & 1 \end{vmatrix} = \begin{pmatrix} B & O \\ U & 1 \end{pmatrix}$$

where the last column is the slack vector associated to the new constraint.

The inverse of the new basis is

$$\bar{B}^{-1} = \begin{pmatrix} B^{-1} & O \\ -UB^{-1} & 1 \end{pmatrix}$$

---

1) See for example: CHAIHO KIM, "Introduction to Linear Programming", Holt, Rinehart and Winston, Inc. 1971, p. 291
2) Hadley, G., op. cit. p. 385

Proof:

$$\begin{pmatrix} B & 0 \\ U & 1 \end{pmatrix} \cdot \begin{pmatrix} B^{-1} & 0 \\ -UB^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let $-UB^{-1} = -\bar{U}$ then,

$$\bar{B}^{-1} = \begin{pmatrix} B^{-1} & 0 \\ -\bar{U} & 1 \end{pmatrix}$$

Now consider the addition of a new "real" variable $x_e$ $e = k+1$ with the vector $P_e$. This is once more, another postoptimal problem where a new variable must be considered.(1) To decide whether to introduce the new variable in the current solution depends on its evaluation relative to the current basis. Using the evaluation rule applied by the RSM, the cost coefficients of the new variable - its profitability chance - is defined through:

$$\beta_e = \beta_r \cdot a_{re} - c_e^{(\S)} \qquad r = 1,2,\ldots,k$$

(Notice that we do not need to consider the cost coefficient of the slack variable since its value is always ZERO)

Let $x_e$ be favorable!

Having determined the inverse of the "current" basis, we merely need to find the value of the new vector in terms of $\bar{B}^{-1}$ through:

---

1) Bloech, Jürgen, "Zum Problem der Nachträglichen Änderung industrieller Produktionsprogramme", op. cit. pp. 186 - 197

§) See page (43)

$$\begin{pmatrix} y_{re} \\ --- \\ y_{ge} \end{pmatrix} = \begin{pmatrix} B^{-1} & 0 \\ -------- & \\ -U & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{re} \\ -- \\ a_{ge} \end{pmatrix} = \begin{pmatrix} B^{-1} \cdot \\ --------- \\ -\bar{U} . a_{re} + a_{ge} \end{pmatrix}$$

It is interesting to notice that the element $y_{ge}$ could be defined through

$$y_{ge} = -\bar{U}\, a_{re} + a_{ge}$$

$$= -U \cdot B^{-1}\, a_{re} + a_{ge} \qquad \text{since } \bar{U} = -U\, B^{-1}$$

Or $$= -U\, y_{re} + a_{ge} \text{ since}$$

$$B^{-1} \cdot a_{re} = y_{re}.$$

This conclusion is very important. Perhaps it is the basic idea of the MKM. The importance of this idea is that we are able to define the value of the new activity $X_e$ in terms of the current basis without being forced to define $\bar{U}$ i.e., the relative coefficients of the hitherto redundant row(s). This means that our matrix, for any reason, is enlarged not only with one constraint but with (m-k) constraints, we do not need to define:

$$-U_d \cdot B^{-1}$$ i.e., the value of the whole annexed constraints in terms of the current solution. Since we are able:

First: to determine the value of the newcoming variable $X_e$ in terms of the current basis through:

a) for the satisfied constraints

$$y_{re} = B^{-1} \cdot a_{re}$$

b) for the redundant constraints (m-k)

$$y_{de} = - U_d \cdot y_{re} + a_{de}$$

$$d = (k+1),(k+2),\ldots,m$$

Second: to determine the value of the outgoing row $\bar{U}$, through

$$\bar{U} = - U \cdot B^{-1}$$

where the outgoing row U is defined, as usual, through the min nonnegative of

$$\bar{p}_i^o / y_{ie} \qquad i = 1,2,\ldots,m.$$

let the min nonnegative be $\bar{p}_g^o / y_{ge}$

Third: to determine the new inverse by pivoting on $y_{ge}$, where the column (e) is the pivot column and the row(g ) is the pivot row.

## 4.8 SUMMARY

Before we treat the MKM coumputationally, it might be convenient to sum up the previous discussion.

Consider the following LP problem.

$$\text{Max.} \quad CX$$

$$\text{subject to} \quad AX \leq b$$

$$\text{and} \quad X \geq 0$$

Suppose we know in advance that the first K real variables $x_1, x_2, \ldots, x_k$ will replace the first K constraints $x_{n+1}, x_{n+2}, x_{n+3}, \ldots, x_{n+k}$.

Our initial tableau is as the following

| | $x_1$ | $x_2$ | $x_k$ | $x_n$ | $x_{n+1}$ | $x_{n+2}$ | $x_{n+k}$ | $x_{n+m}$ | |
|---|---|---|---|---|---|---|---|---|---|
| $x_{n+1}$ | $a_{11}$ | $a_{12}$ | $a_{1k}$ | $a_{1n}$ | 1 | 0 | 0 | 0 | $p_1^o$ |
| $x_{n+2}$ | $a_{21}$ | $a_{22}$ | $a_{2k}$ | $a_{2n}$ | 0 | 1 | 0 | 0 | $p_2^o$ |
| | ... | | | ... | | | ... | ... | |
| $x_{n+k}$ | $a_{k1}$ | $a_{k2}$ | $a_{kk}$ | $a_{kn}$ | 0 | 0 | 1 | 0 | $p_k^o$ |
| | ... | | ... | | ... | | | ... | |
| $a_{n+m}$ | $a_{m1}$ | $a_{m2}$ | $a_{mk}$ | $a_{mn}$ | 0 | 0 | 0 | 1 | $p_m^o$ |
| | $c_1$ | $c_2$ | $\ldots c_k \ldots$ | $c_n$ | 0 | 0 | 0 | ... 0 | |

Finding directly the inverse of $a_{rr}$ $r = 1,2,\ldots,k$; $a_{rr}^{-1}$, the value of the current solution is defined through:

$$\bar{p}_r^o = a_{rr}^{-1} \cdot p_r^o$$

The value of the cost coefficients $\beta_r$ $r = 1,2,\ldots,k$ is defined through:

$$\beta_r = c_r \cdot A_{rr}^{-1}$$

The value of the redundant capacity is defined through:

$$\bar{p}_d^o = p_d^o - a_{dr} \cdot \bar{p}_r^o$$

$$r = 1,2,\ldots,k$$
$$d = (k+1),(k+2),\ldots,m$$

Our tableau would be as follows:

| | $x_1$ | $x_2$ | $\ldots x_k$ | $x_{k+1}\ldots x_n$ | $b_1$ | $b_2\ldots b_k$ | $b_{k+1}\ldots b_m$ | |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 0 | $a_{1,k+1}\ldots a_{1n}$ | $y_{11}$ | $y_{12}$ $y_{1k}$ | 0 ... 0 | $\bar{p}_1^o$ |
| $x_2$ | 0 | 1 | 0 | $a_{2,k+1}\ldots a_{2n}$ | $y_{21}$ | $y_{22}$ $y_{2k}$ | 0 ... 0 | $\bar{p}_2^o$ |
| $x_k$ | 0 | 0 | 1 | $a_{k,k+1}\ldots a_{kn}$ | $y_{k1}$ | $y_{k2}$ $y_{kk}$ | 0 ... 0 | $\bar{p}_k^o$ |
| $x_h$ | $a_{h1}$ | $a_{h2}$ | $a_{hk}$ | $a_{h,k+1}\ldots a_{hn}$ | 0 | 0 | 1 0 | $\bar{p}_h^o$ |
| $\vdots$ $x_{n+m}$ | $a_{m1}$ | $a_{m2}$ | $a_{mk}$ | $a_{m,k+1}\ldots a_{mn}$ | 0 | 0 | 0 0 1 | $\bar{p}_m^o$ |
| | 0 | 0 | 0 | $c_{k+1}\ldots c_n$ | $\beta_1$ | $\beta_2$ $\beta_k$ | ... 0 ... 0 | |

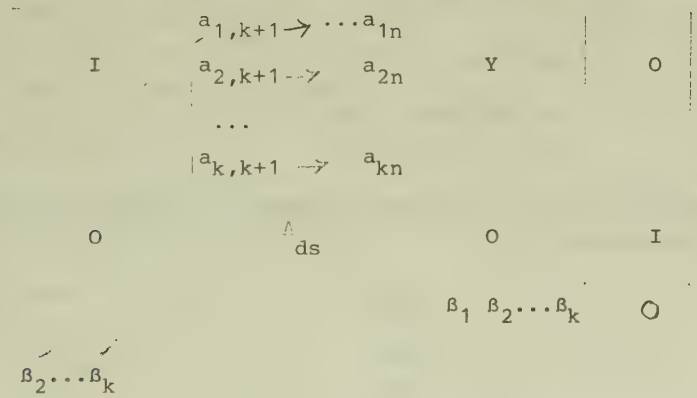
where $h = k+1$

(notice that the symbol (a) means the original element at the initial tableau while the symbol (y) means an element of the inverse $A_{rr}^{-1}$ ).

To move to an adjacent vertix, the cost coefficients $\beta_s$ $s = k+1,k+2,\ldots,n$ must be determined through:

$$\beta_s = \beta_r \cdot A_{rs} - c_s$$

(note that this is a relation between the current cost coefficients and the original data of the problem or more precisely a part of the original data as shown in the following figure: Where the dashed lines connect those numbers that are multiplied, while the arrows indicate the movement of the process when going from one non-basic "real" variable to another




Repeating this procedure for each non-basic variable we find $\beta_e$ by :

$$\beta_e = \min \beta_j \qquad j = 1,2,\ldots,n$$

Determination of $y_{ie}$ $\qquad i = 1,2,\ldots,m$

If the variable $x_e$ is to be introduced, we must find the next constraint to be exhausted and this needs a knowledge of $y_{ie}$

i) we first find $y_{re}$ $r = 1,2,\ldots,k$ . This is done by multiplying the subvector $a_{re}$ with the inverse of the current solution $a_{rr}^{-1}$.

$$y_{re} = a_{rr}^{-1} \cdot a_{re}$$

ii) Secondly, we define $y_{de}$ through

$$y_{de} = a_{de} - a_{dr} \cdot y_{re}$$

$$d = k+1,k+2,\ldots,m$$
$$r = 1,2,\ldots,k$$

The evaluation is indicated schematically in the following figure:

$$\begin{matrix} y_{1e} \\ y_{2e} \\ y_{ke} \end{matrix}$$

$$\begin{pmatrix} a_{k+1,1} & a_{k+1,1} & \cdots a_{k+1,k} \end{pmatrix}$$

where the dashed lines connect those numbers that are multiplied, while the arrows indicate the movement when moving from one vextor to another.

Finding $y_{ie}$ $i = 1,2,\ldots,m$ in this manner, we can determine the constraint to be exhausted through determining:

$$\text{the min non-neg.} \quad \bar{p}_i^o \; / \; y_{ie}$$

Let it be $p \overset{o}{g} / y_{ge}$

Having defined the outgoing row, we need to define its value in terms of the current basis $a_{rr}^{-1}$ through:

$$y_{gr} = -a_{gr} \cdot a_{rr}^{-1} \qquad r = 1,2,\ldots,k$$

To end this iteration, we need to perform the usual transformation pivoting on $y_{ge}$.

The elements to be transformed are:

1. The inverse $A_{rr}^{-1}$

$$A'^{-1}_{rr} = A_{rr}^{-1} - y_{re} \cdot y_{gr} / y_{ge}$$

2. The subvector $y_{re}$

$$y'_{re} = - y_{re} / y_{ge}$$

3. The subvector $y_{gr}$ and the element $y_{ge}$

$$y'_{gr} = y_{gr} / y_{ge}$$

$$y'_{ge} = 1/y_{ge}$$

4. The subvector $\beta_r$, and the element $\beta_e$

$$\beta'_r = \beta_r - \beta_e \cdot y_{gr} / y_{ge}$$

$$\beta'_e = \beta_e / y_{ge}$$

5. The constant vector $p_i^o$

$$\grave{p}_i^o = \bar{p}_i^o - y_{ie} \cdot \bar{p}_g^o / y_{ge} \qquad i \neq g$$

$$\grave{p}_g^o = \bar{p}_g^o / y_{ge}$$

## 5. THE MODIFIED KAIRO METHOD-COMPUTATIONALLY.

In this chapter we shall apply the theory developed in the preceding chapter to the solution of actual problems. This involves especially the discussion of the various aspects of the computational procedure. Additionally, we shall answer some specific questions concerning the role of the slack variables.

At the outset of our discussion, we would like to stress once more the fact that the KM and its modified form are not suitable at all for hand computation even for problems of moderate dimensions. Hence we are not going to suggest a standard tableau for them. All our concern will be devoted to prove their compatibility to the execution on the digital computers.

It is easy to imagine why the KAIRO algorithm is rather troublsome;to solve systems simultaneously by hand is very limited, and in both the KM and its modified form arises the difficulty of finding the elements to be solved simultaneously or to be transformed in the case of the KM or the MKM respectively, when the variables introduced and the constraints satisfied would not necessarily follow as neat a form as assumed in our discussion.

### . THE ROLE OF THE SLACK VARIABLES.

Up to the present time, all our discussions have been based on the assumption that a system of slack variables has been annexed to the original problem. "Basically, finding an initial solution means the selection of a set of m independent vectors to form a basis B so that $(B^{-1} \cdot p^{o})$ yields an immediate unique solution".(1)

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1) An-min Chung, op. cit. p. 125

Another job of the slack variables is discussed by Heeady and Candler.[1] They asked: "Can these columns-slack variables- be left out of the simplex tableau to lesson the computation?" They answered this question in the following manner: "Slack variables can be left out if we are certain that the activities will come into successive programs in a particular manner: activities must come into the program to just exactly exhaust resource supplies. No disposal activities must be reduced to zero, to be later increased or returned to a non-zero level ... if these conditions will not hold true, we should include the disposal activities in the computational tableau. They gave the following example to show the role of the slack variables:

"Suppose a programming problem has two resource restrictions, represented by $ 200 of capital and 12 hours of labor. The two real activities are: corn which uses $ 20 of capital and 2 hours of labor per acre, and wheat which uses $ 10 of capital and 5 hours of labor per acre. Omitting columns for disposal activities, the simplex form is that of table 3.5 where the columns and rows are those explained earlier.

Table 3.5. Simplex Table Without Disposal Columns

| | | C → | 40 | 30 | |
|---|---|---|---|---|---|
| | | | Real Activities | | R |
| $C_s$ | Resource or activity at non-zero level | Resource supply or activity level B | Corn $P_1$ (acre) | Wheat $P_2$ (acre) | |
| 0 | Capital $P_3$ | 200 | 20 | 10 | 10 |
| 0 ← | Labor $P_4$ | 12 | (2) | .5 | 6 |
| | Z | 0 | 0 | 0 | |
| | Z - C | 0 | [-40] | -30 | |
| 0 ← | Capital $P_3$ | 80 | 0 | (5) | 16 |
| 40 → | Corn $P_4$ | 6 | 1 | .25 | 24 |
| | Z | 240 | 40 | 10 | |
| | Z - C | 240 | 0 | [-20] | |
| 30 → | Wheat $P_2$ | 16 | 0 | 1 | |
| 40 | Corn $P_1$ | 2 | 1 | 0 | |
| | Z | 560 | 40 | 30 | |
| | Z - C | 560 | 0 | 0 | |

[1] Heady,E., and Candler,W., op cit pp. 78-81

Using the criteria and computational procedures already explained, corn first comes into the plan at the level, determined by labor, of 6 acres. Capital is reduced to 80 as indicated in the second tableau. However, the program with 6 acres of corn is not optimum because the $z_j - c_j$ value for wheat is - 20 in the second tableau. With an R of 16, capital is the most limiting resources for wheat. A third plan is therefore formed in the third tableau and includes 16 acres of wheat and 2 acres of corn. Corn is reduced to 2 acres in order that labor can be released for 16 units of wheat. However, without disposal activities, this plan is indicated as optimum since the final Z - C row contains no negative figures.

The same problem is now programmed in table 3.6, where disposal activities are included for capital and labor. However, this table now includes four sections or programs, in contrast to three programs for table 3.5. Another program has been computed, and its profit of $ 600 is greater than the $ 560 of table 3.5.

Using the conventional computational procedures, and aside from the disposal columns, programs in table 3.6. are the same as those in table 3.5 through the third tableau. However, the Z - C row in the third tableau of table 3.6 has a - 20 for labor disposal, indicating that this activity should be restored to the program at a non-zero level.(1)

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(1) Heady O. E., and Candler W., "Linear Programming Methods", The Iowa State University, USA, 1958 pp. 75-81

Table 3.6. Simplex Table With Disposal Columns

| | | C → | 0 | 0 | 40 | 30 | |
|---|---|---|---|---|---|---|---|
| | | | Disposal activities | | Real activities | | |
| $C_s$ | Resource or activity at non-zero level | Resource supply or activity level B | Capital disposal $P_3$ | Labor disposal $P_4$ | Corn $P_1$ (acre) | Wheat $P_2$ (acre) | R |
| 0 | Capital $P_3$ | 200 | 1 | 0 | 20 | 10 | 10 |
| 0 ← | Labor $P_4$ | 12 | 0 | 1 | (2) | .5 | 6 |
| | Z | 0 | 0 | 0 | 0 | 0 | |
| | Z - C | 0 | 0 | 0 | [-40] | -30 | |
| 0 ← | Capital $P_3$ | 80 | 1 | -10 | 0 | (5) | 16 |
| 40 → | Corn $P_1$ | 6 | 0 | .5 | 1 | .25 | 2 |
| | Z | 240 | 0 | 20 | 40 | 10 | |
| | Z - C | 240 | 0 | 20 | 0 | [-20] | |
| 30 → | Wheat $P_2$ | 16 | .2 | -2 | 0 | 1 | unlimited |
| 40 ← | Corn $P_1$ | 2 | -.05 | (1) | 1 | 0 | 2 |
| | Z | 560 | 4 | -20 | 40 | 30 | |
| | Z - C | 560 | 4 | [-20] | 0 | 0 | |
| 30 | Wheat $P_2$ | 20 | .1 | 0 | 2 | 1 | |
| 0 → | Labor $P_4$ | 2 | -.05 | 1 | 1 | 0 | |
| | Z | 600 | 3 | 0 | 60 | 30 | |
| | Z - C | 600 | 3 | 0 | 20 | 0 | |

Computing the R values, we see that labor disposal should replace corn in the final program. Hence, the corn row is replaced by the labor row in section four. However, at the same time, wheat is increased from 16 to 20 units, as labor disposal is brought into the final program at a level of 2. Wheat is increased by 4 units since the coefficient of labor disposal for wheat is - 2 in the third tableau. By including labor disposal, we allow the - 2 coefficient to come into the tableau. The - 2 coefficient gives rise to $z_j - c_j = -20$ for labor disposal in the third tableau for this reason: An increase in labor disposal reduces by 1 acre the amount of corn produced (indicated by 1 in the labor disposal column and corn row) and increases by 2 acres the amount of wheat

produced (indicated by - 2 in the labor disposal column and wheat row). The value of the 1 acre of corn reduced is $ 40; the value of the 2 acres of wheat added is $ 60; this gives a net gain of $ 20, which appears, of course, as - 20 in the Z - C row.

Without allowing this indirect evaluation of labor disposal, as a means of reducing corn to be replaced with the more profitable wheat activity, it was necessary to stop with the third, or sub-optimum, program in table 3.6. No method existed for eliminating corn from the plan and allowing labor disposal to re-enter at a non-zero level, in order to allow a further increase in wheat.

It is interesting to indicate here, that both of the KM and the MKM can eliminate corn from the plan and allowing labor disposal to re-enter at a non-zero level without using slack-variables.

The first iteration:

| $x_1$ | $x_2$ | |
|---|---|---|
| 20 | 10 | 200 |
| 2 | 0,5 | 12 |
| -40 | -30 | |

Deciding to introduce $x_1$ to replace $b_2$, the inverse at the first iteration is

$$A^{-1}_{zq} = 1/a_{21} = 0,5 \qquad \begin{matrix} z = 2 \\ q = 1 \end{matrix}$$

$$p'^{o}_{z} = A^{-1}_{zq} \cdot p^{o}_{z} = 6$$

$$b_d = p'^{o}_{d} = p^{o}_{d} - a_{dq} \cdot p^{o}_{q} \qquad d = 1$$

$$= 20 - (20 \cdot 6) = 80$$

$$\beta_z = c_q \cdot A^{-1}_{zq} \qquad \begin{matrix} z = 2 \\ q = 1 \end{matrix}$$

$$= 40 \cdot 0,5 = 20$$

$$f(x) = c_q \cdot p'^{o}_{z} = 40 \cdot 6 = 240$$

The second iteration

$\beta_s$ is defined through:

$$\beta_s = \beta_z \cdot a_{zs} - c_s \qquad s = 2$$

$$= 20 \cdot 0,5 - 30 = -20$$

Then $x_2$ must enter the solution, $y_{i2}$ is defined through:

$$y_{z2} = A^{-1}_{zq} \cdot a_{z2} \qquad z = 2$$

$$= 0,5 \cdot 0,5$$

$$= 0,25$$

$$y_{d2} = a_{d2} - a_{dq} \cdot y_{z2} \qquad d = 1$$

$$= 10 - 20 \cdot 0,25$$

$$= 5$$

Now $p'^{o}_{i} / y_{i2}$ must be defined.

$$p'^{o}_{1} / y_{12} = 80/5 = 16$$

$$p'^{o}_{2} / y_{22} = 6 / 0{,}25 = 24$$

The inverse of the current basis is

$$A^{-1}_{zq} = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}^{-1} \qquad \begin{matrix} z = 2,1 \\ q = 1,2 \end{matrix}$$

$$= \begin{pmatrix} 1 & -0{,}05 \\ -2 & 0{,}2 \end{pmatrix}$$

$$\beta_z = c_q \cdot A^{-1}_{zq} \qquad \begin{matrix} z = 2,1 \\ q = 1,2 \end{matrix}$$

$$\begin{bmatrix} \beta_2 & \beta_1 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \cdot A^{-1}_{zq}$$

$$\beta_2 \quad \beta_1 = (40 \quad 30) \cdot \begin{pmatrix} 1 & -0{,}05 \\ -2 & 0{,}2 \end{pmatrix}$$

$$= (-20 \quad 4)$$

The constant column is defined through:

$$\text{First } p'^{o}_{z} = \begin{pmatrix} p'^{o}_{2} \\ p'^{o}_{1} \end{pmatrix} = A^{-1}_{zq} \cdot p^{o}_{z}$$

$$= \begin{pmatrix} 1 & -0{,}05 \\ -2 & 0{,}2 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 200 \end{pmatrix} = \begin{pmatrix} 2 \\ 16 \end{pmatrix}$$

Second $p'^{o}_{d}$ - we may notice here that no need to define $p'^{o}_{d}$ since d = 0, in other words, no reduncant capacity.

$$f(x) = x_q\, p^{o}_{z} = \begin{pmatrix} 40 & 30 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 16 \end{pmatrix} = 560$$

The third iteration

An imaginary tableau after the third iteration would be as follows:

| | $b_2$ | $b_1$ | |
|---|---|---|---|
| $x_1$ | 1 | -0,05 | 2 |
| $x_2$ | -2 | 0,2 | 16 |
| | -20 | 4 | |

(notice the interchange of the indices b and x).

Deciding to introduce $b_2$ to the basis means that $b_2$, after being reduced to zero, is going to assume a non-zero level.

To determine the outgoing variable we find the min nonnegative of

$$2/1 = 2 \qquad \frac{16}{-2} = -8$$

Pivoting on $y_{11}$, and neglecting the first column and the first row, the tableau would be as follows (§)

| | $x_1$ | $b_1$ | |
|---|---|---|---|
| $b_2$ | | | 2 |
| $x_2$ | | 0,1 | 20 |
| | | 3 | |

$f(x) = 20 \cdot 30 = 600$

In each a case the missing row and vector must be re-read as follows:

| | $x_1$ | $b_1$ | |
|---|---|---|---|
| $b_2$ | 2 | 0,5 | 2 |
| $x_2$ | 20 | 0,1 | 20 |
| | -40 | 3 | |

to examine the possibility of inserting a new variable, we compute $\beta_s$, s = 1

$$\beta_s = \beta_z \cdot a_{zs} - c_s$$

$$= 3 \cdot 20 - 40$$

$$= 20$$

Since $\beta_s$ is positive-stop, the current solution is optimal.

To explain how to deal with such cases, a more complicated example will be solved later using the MKM.

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(§) see pp. (141-148 ) this point is discussed in detail.

## 5 .2 LINEAR PROGRAMMING WITHOUT SLACK VARIABLES.

The KM and its modified form do not require slack variables as an initial basic feasible solution or when the activities return to a non-zero level. The computation starts with a one-real-variable as an initial "degenerated" basic feasible solution. In the case of maximization we start - as usual - introducing the variable which yields the greatest cost coefficient, as long as it is feasible, that is, when at least one of the ratios $p'^{o}_{i} / y_{ie}$ (§) $i = 1,2,\ldots,m$ is of non-negative value.

Certainly it is more convenient to work with equations rather than inequalities. (1) But once we are not going to solve the whole problem in one iteration - indeed we are not allowed to - and the optimal solution, assuming that one exists, would be defined through a "long" iterative process by satisfying the constraints one after another, it is not necessary to convert the inequalities into equalities. The difference between the amount available and the amount actually used would be "redundant" capacity rather than a pseudo variable. The case when the activities returned to a nonzero level is the subject matter of the next discussion. First we will discuss some new properities of the inverse of a matrix.

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1) Hadley, G., op. cit. p. 72

§) At the first iteration $y_{ie} = a_{ie}$

## 5.3 Some new properties of the inverse of a matrix: (§)

Before discussing the question of the returning constraints, it might be interesting to indicate here that the Kairo Algorithm for matrix inversion reveals some new properties of the inverse of a matrix.

Property 1: If $A^{-1}$ the inverse of A

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{1m} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{im} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1m} \\ y_{21} & y_{22} & \cdots & y_{2m} \\ \vdots & & & \\ y_{i1} & y_{i2} & \cdots & y_{im} \\ \vdots & & & \\ y_{m1} & y_{m2} & \cdots & y_{mm} \end{pmatrix}$$

Pivoting on $y_{ij}$ will give the inverse $A'^{-1}$ where A' is the same as A without the jth row and the ith column. §§

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(§) In the available literature, these properties have never been discussed.

(§§) A proof for this property has been already dicussed on pages 112-114

For instance if

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$$

Pivoting on $y_{12}$ , for example, the resultant submatrix, neglecting the pivot row and the pivot cloumn would be

$$B^{-1} = \begin{pmatrix} y'_{21} & y'_{23} \\ y'_{31} & y'_{33} \end{pmatrix}$$

where

$$B = \begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix}$$

Considering the same matrix inverted by Kairo Algorithm page (101)where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 4 \\ 1 & 3 & 2 \end{pmatrix}$$

and

$$A^{-1} = \begin{bmatrix} 4/9 & 1/9 & -2/9 \\ 1/9 & -2/9 & 4/9 \\ -7/18 & 5/18 & 1/18 \end{bmatrix}$$

pivoting on the circled element ($y_{12}$), and then neglecting the first row and the second column, the resultant submatrix would be

$$B^{-1} = \begin{bmatrix} 1 & 0 \\ -3/2 & 1/2 \end{bmatrix}$$

where

$$B = \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

Proof: $B^{-1} \cdot B = I$

$$\begin{bmatrix} 1 & 0 \\ -3/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Property 2: If

$$A = \begin{pmatrix} a_{11} & a_{1g} & a_{1m} \\ \vdots & & \\ a_{i1} & a_{ig} & a_{im} \\ \vdots & & \\ a_{m1} & a_{mg} & a_{mm} \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} y_{11} & y_{1g} & y_{1m} \\ \vdots & & \\ y_{i1} & y_{ig} & y_{im} \\ \vdots & & \\ y_{m1} & y_{mg} & y_{mm} \end{pmatrix}$$

and $y_{ie} = A^{-1} \cdot a_{ie}$ $\quad i = 1,2,\ldots,m.$

Pivoting on $y_{ge}$ and neglecting the column $y_{ie}$ - the resultant matrix would be $B^{-1}$, where B is

$$B = \begin{pmatrix} a_{11} & a_{1e} & a_{1m} \\ \vdots & & \\ a_{i1} & a_{ie} & a_{im} \\ \vdots & & \\ a_{m1} & a_{me} & a_{mm} \end{pmatrix}$$

(Notice that the index e replaced the index g)

Let $$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

and $$A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

and $$y_{ie} = A^{-1} \cdot a_{ie} \qquad i = 1,2$$

$$\begin{bmatrix} -4 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

pivoting on $y_{1e}$

$$\left[\begin{array}{cc|c} 2 & -1 & -4 \\ -3 & 2 & 8 \end{array}\right]$$

The resultant matrix $B^{-1}$

$$B^{-1} = \begin{bmatrix} -1/2 & 1/4 \\ 1 & 0 \end{bmatrix}$$

where $B = \begin{bmatrix} a_{1e} & a_{12} \\ a_{2e} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}$

Proof $B^{-1}B = I$

$$\begin{bmatrix} -1/2 & 1/4 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Property 3:

If

$$A = \begin{pmatrix} a_{11} & a_{1i} & a_{1m} \\ \vdots & & \\ a_{e1} & a_{ei} & a_{em} \\ \vdots & & \\ a_{m1} & a_{mi} & a_{mm} \end{pmatrix}$$

And

$$A^{-1} = \begin{pmatrix} y_{11} & y_{1i} & y_{1m} \\ \vdots & & \\ y_{e1} & y_{ei} & y_{em} \\ \vdots & & \\ y_{m1} & y_{mi} & y_{mm} \end{pmatrix}$$

and $y_{gi} = - a_{gi} \cdot A^{-1}$ $\qquad i = 1,2,\ldots,m$

Pivoting on the $y_{ge}$ element and neglecting the row vector $y_{gi}$, the resultant matrix is $B^{-1}$, where B

$$B = \begin{pmatrix} a_{11} & a_{1e} & a_{1m} \\ \vdots & & \\ a_{g1} & a_{ge} & a_{gm} \\ \vdots & & \\ a_{m1} & a_{me} & a_{mm} \end{pmatrix}$$

(notice the intercharge between the indices e and g).

Let $$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

and $$A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

and $$y_{gi} = - [a_{gi}] \cdot [A^{-1}]$$

$$[7 - 5] = - [1 \quad 3] \cdot \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Pivoting on, say, $y_{g2}$,

$$\begin{vmatrix} 2 & -1 \\ -3 & 2 \\ 7 & -5 \end{vmatrix}$$

neglecting the row $y_{gi}$, the resultant matrix $B^{-1}$ would be:

$$B^{-1} = \begin{bmatrix} \frac{3}{5} & -1/5 \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

where

$$B = \begin{bmatrix} a_{11} & a_{12} \\ a_{g1} & a_{g2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Proof $B^{-1}B = I$

$$\begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

## 5.4 Further discussion

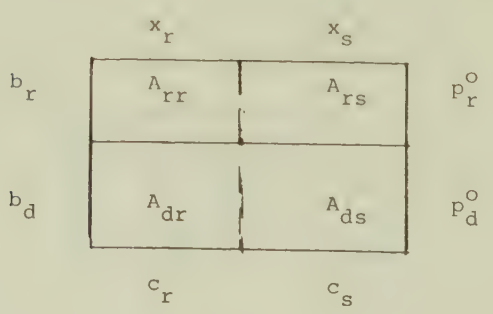
Consider the following LP problem

$$\max \quad C_j x_j$$

$$\text{subject to} \quad A_{ij} x_j \leq p_i^o$$

$$\text{and} \quad x_j \geq 0$$

The problem in a tableau form would be as follows:

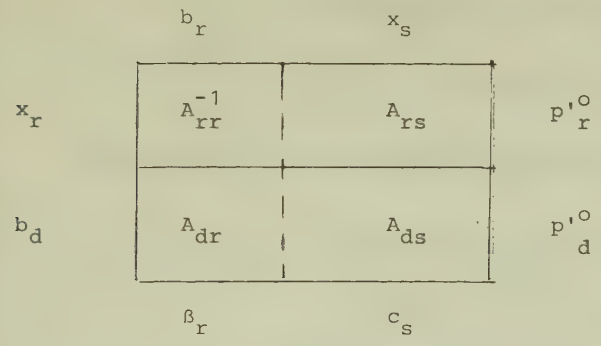



where $r = 1,2,\ldots,k$

$s = k+1,k+2,\ldots,n$

$d = k+1,k+2,\ldots,m$

Let us assume that after k iteration, the first k real variables replaced the first k constraint. Applying the MKM we would have the following tableau:




(observe the interchange between the index vectors $x_r$ and $b_r$).

It must have been clear that moving towards an adjacent extreme point turns to be a matter of:

1. Defining a new pivot element $y_{ge}$

2. Updating the tableau to obtain the inverse of the new basis.

During our previous discussion, we have assumed that the pivot element is $y_{ge}$ ; where g is an element of d; d = k+1, k+2,...,m - and e is an element of s ; s = k+1,k+2,...,n.

This, however, not

The operator may be faced with one of the following cases:

1. The largest negative relative cost coefficient is $\beta_e$, where the index e is:

   a) an element of s, $s = k+1,k+2,\ldots,n$

      or

   b) an element of r, $r = 1,2,\ldots,k$

2. The minimum nonnegative ratio is $p_g^o / y_{ge}$ where the index g is:

   a) an element of d, $d = k+1,k+2,\ldots,m$

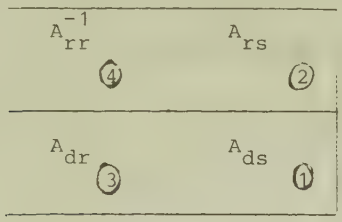
      or

   b) an element of r, $r = 1,2,\ldots,k$

In other words, the pivot element could be any element of the whole tableau:

| | |
|---|---|
| $A_{rr}^{-1}$ $(y_{ge})$ | $A_{rs}$ $(y_{ge})$ |
| $A_{dr}$ $(y_{ge})$ | $A_{ds}$ $(y_{ge})$ |

To simplify the presentation let us numerate the different four cases as follows:




We will discuss now how the vector $y_{ie}$, i.e., the value of the newcoming column in terms of the basis and the vector $y_{gr}$, i.e., the value of the outgoing row in terms of the current inverse are computed. We will show, as well, how to update the tableau in each case making use of the new properites of the inverse of a matrix previously presented.

Case 1 Having defined $\beta_e$, where e is an element of s, $s = k+1,k+2,\ldots,n$ - $y_{ie}$ is defined as follows:

First $\quad y_{re} = A_{rr}^{-1} \cdot a_{re}$

and then $\quad y_{de} = a_{de} - A_{dr} \cdot y_{re}$

where $\quad r + d = m$

the pivot row is defined through:

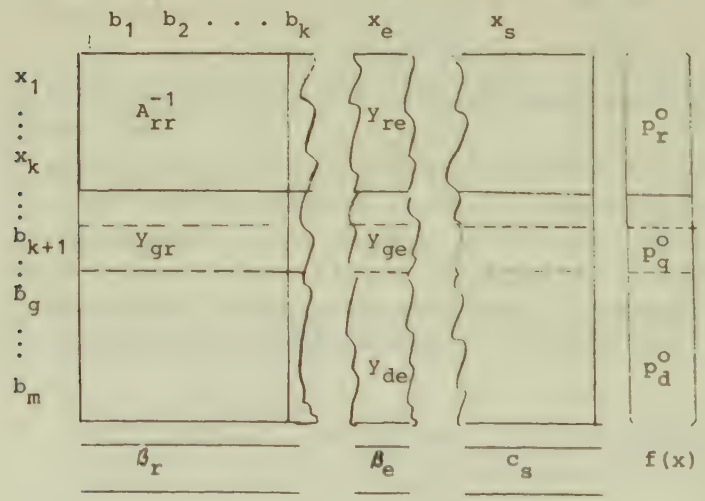
$p_g^o / y_{ge}$ = the min. nonnegative $p_i^o / y_{ie}$

where g is an element of d, $d = k+1,k+2,\ldots,m$

The vector $y_{gr}$ is defined through:

$$y_{gr} = - \; a_{gr} \cdot A_{rr}^{-1} \qquad r = 1,2,\ldots,k$$

An imaginary tableau would be as follows:




To end this iteration, we perform the normal transformation process pivoting on $y_{ge}$. Only the following elements must be transformed.

1. The elements of the inverse $A_{rr}^{-1}$

$$A'^{-1}_{rr} = A_{rr}^{-1} - y_{re} \cdot y_{gr} \; / \; y_{ge}$$

2. The subvector $y_{re}$

$$y'_{re} = - \; y_{re} \; / \; y_{ge}$$

3. The subvector $y_{gr}$ and the element $y_{ge}$

$$y_{gr} = y_{gr} \; / \; y_{ge}$$

$$y_{ge} = 1 \; / \; y_{ge}$$

4. The subvector $\beta_r$, and the element $\beta_e$

$$\beta_r = \beta_r - \beta_e \cdot y_{gr} \; / \; y_{ge}$$

$$\beta_e = \beta_e \; / \; y_{ge}$$

5. The constant vector $p_i^o$

$$p_i^o = p_i^o - y_{ie} \cdot p_g^o \; / \; y_{ge} \qquad i \cdot g$$

$$p_g^o = p_g^o \; / \; y_{ge}$$

The dimensions of the new inverse will increase to k+1, the basic variables would be $(x_1, x_2 \ldots, x_k, \underline{x_e})$, the exhausted constraints would be $(b_1, b_2, \ldots, b_k, \underline{b_g})$. In short the new inverse $A^{-1}$ will be the inverse of the following system:

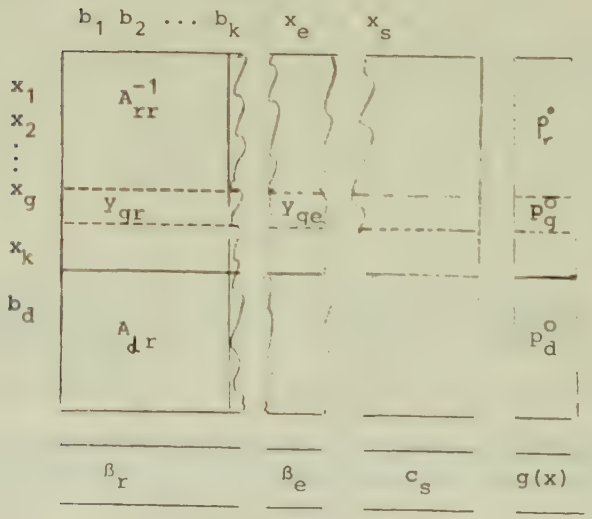
$$\begin{bmatrix} a_{11} & a_{12} \cdots a_{1k} & a_{1e} \\ a_{21} & a_{22} \cdots a_{2k} & a_{2e} \\ \vdots & & \\ a_{k1} & a_{k2} \cdots a_{kk} & a_{ke} \\ a_{g1} & a_{g2} \cdots a_{gk} & a_{ge} \end{bmatrix}$$

Case 2 : Having defined $\beta_e$, where e is an element of s, we define $y_{ie}$ as shown in case 1.

$$\bar{p}^o_g / y_{ge} = \text{the min. nonnegative } \bar{p}^o_i / y^o_{ie}$$

where g is an element of r, r = 1,2,...,k.

This means that the g+h basic variable must leave the basis to be replaced with $x_e$ - an imaginary tableau would be as follows:




Pivoting on $y_{ge}$ and transforming only the following elements:

1. The elements of the inverse $A^{-1}_{rr}$

$$A'^{-1}_{rr} = A^{-1}_{rr} - y_{re} \cdot y_{gr} / y_{ge}$$

where the row vector is transformed as follows

$$y'_{gr} = y_{gr} / y_{ge}$$

2. The subvector $\beta_r$

$$\beta'_r = \beta_r - \beta_e \cdot y_{gr} / y_{ge}$$

3. The constant vector $p_i^o$

$$p_i^o = p_i^o - y_{ie} \cdot p_g^o / y_{ge} \qquad i \neq g$$

$$p_g^o = p_g^o / y_{ge}$$

The basic variables are now

$$(x_1, x_2, \ldots, x_e, \ldots, x_k) \quad \ldots. \quad \text{(porperty 2)};$$

they were

$$(x_1, x_2, \ldots, x_g, \ldots, x_k)$$

The constraints being exhausted are still the same as before $(b_1, b_2, \ldots, b_k)$

In such a case, the original subvector $a_{rg}$ together with $c_g$ must be re-read once more to be annexed to the subvector $a_{dg}$ - since $x_g$ might prove later to be favorable.

Case 3: When $\beta_e$ = the largest negative $\beta_j$ where e is an element of r , r = 1,2,...,k and $y_{ge}$ = min. non-negative $p_i^o / y_{ge}$

where g is an element of d, d = k+1,k+2,...,m .

Such a "case" which is meant by Heady and Candler and which necessistates the inversion of the slack variables. In such a case, the constraint having the

index g , will be reduced to zero.

Let us now see how we could further work on the problem in the absence of the slack variable. Here we don't need to find the value of the newcoming vector in terms of the basis, since the subvector $y_{re}$ is already a part of the inverse $A_{rr}^{-1}$ . We need, however, to define the sub-column vector $y_{de}$ , d = k+1,...,m. In other words we want to determine the subcolumn vector $y_{d,k+e}$, if a slack variable for the constraint $b_e$ were attached and computed - as in the standard simplex procedure.

Recalling the simplex tableau Tsim at the k+h iteration

| | | | |
|---|---|---|---|
| I | $A_{rr}^{-1} \cdot A_{rs}$ | $A_{rr}^{-1}$ | O |
| O | $A_{ds} - A_{dr} \cdot a'_{rs}$ | $-A_{dr} \cdot A_{rr}^{-1}$ | I |

it might be clear that the missing subcolumn vector $y_{de}$ is a column vector of the submatrix

$$- A_{dr} \cdot A_{rr}^{-1}$$

which is the negative product of

$$\begin{vmatrix} a_{k+1,1} & \cdots & a_{k+1,k} \\ \vdots & & \\ a_{m1} & \cdots & a_{mk} \end{vmatrix} \cdot \begin{vmatrix} y_{11} & y_{1k} \\ & \\ y_{k1} & y_{kk} \end{vmatrix}$$

But we are not interested in the whole submatrix $-A_{dr} \cdot A_{rr}^{-1}$. We are interested only in the subvector $y_{de}$ which is the negative product of:

$$\begin{vmatrix} a_{k+1,1} & \cdots & a_{k+1,k} \\ & & \\ a_{m,1} & \cdots & a_{mk} \end{vmatrix} \cdot \begin{vmatrix} y_{1e} \\ \\ y_{ke} \end{vmatrix}$$

which is a relation between the subcolumn vector $y_{re}$ which is a column vector of the already defined $A_{rr}^{-1}$, namely the eth columnvector, and the original submatrix $A_{dr}$. Both of them are available at any iteration.

It is interesting to note that the last step could be accomplished if we consider the newcoming "variable" as to be a unit vector of m order where the unit element is in the eth position.

$$y_{re} = A_{rr}^{-1} \cdot I_e$$

$$y_{de} = a_{de} - A_{dr} \cdot y_{re}$$

$$= - A_{dr} \cdot y_{re} \qquad \text{since } a_{de} = 0$$

To determine $y_{gr}$, the value of the outgoing row on terms of the basis, we find:

First: $\bar{p}_g^o / y_{ge}$, the min nonnegative of $\bar{p}_i^o / y_{ie}$

Second: we multiply $-a_{gr}$ by the inverse $A_{rr}^{-1}$

$$y_{gr} = - a_{gr} \cdot A_{rr}^{-1} \qquad r = 1,2,\ldots,k$$

To end this iteration, we transform only the following elements pivoting on $y_{ge}$

1. The elements of the inverse $A_{rr}^{-1}$

$$A'^{-1}_{rr} = A^{-1}_{rr} - y_{re} \cdot y_{gr} / y_{ge}$$

2. The subvector $y_{re}$

$$y'_{re} = - y_{re} / y_{ge}$$

3. The subvector $\beta_r$ and the element $\beta_e$

$$\beta'_r = \beta_r - y_{gr} \cdot \beta_e / y_{ge} \qquad r \neq e$$

$$\beta'_e = \beta_e / y_{ge}$$

4. The constant vector $p_i^o$

$$\bar{p}_i^o = p_i^o - y_{ie} \cdot p_g^o / y_{ge} \qquad i \neq g$$

$$\bar{p}_g^o = p_g^o / y_{ge}$$

The basic variables would be still the same as before

$$(x_1, x_2, \ldots, x_k)$$

The constraint being exhausted will be

$$(b_1, b_2, \ldots, b_e, \ldots, b_k)$$

They were

$$(b_1, b_2 m, \ldots, b_g, \ldots b_k) \ldots\ldots\ldots \text{ property 3}$$

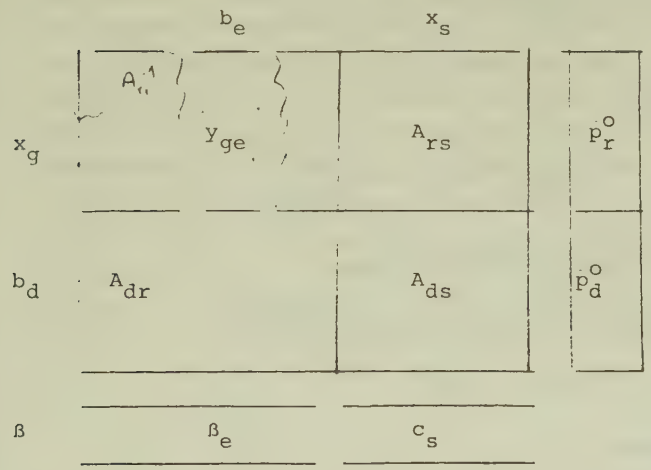
In such a case, the original subvector $a_{er}$ must be re-read once more to be annexed to the subvector $a_{es}$ - otherwise the "returned" constraint would be neglected.

Case 4: Having defined $\beta_e$, where e is an element of r, r = 1,2,...,k and the columnvector $y_{ie}$ as discussed in case 3, the ratio

$$\bar{p}^o_g / y_{ge} = \text{min nonnegative } \bar{p}^o_i / y_{ie}$$

where g is an element of r , r = 1,2,...,k

An imaginary tableau would be as follows:




We end this iteration by pivoting on $y_{ge}$ and performing the normal transformation procedure only for the following elements:

1. The elements of the inverse $A^{-1}_{rr}$

$$A'^{-1}_{rr} = A^{-1}_{rr} - y_{re} - y_{gr} / y_{ge}$$

2. The subvector $\beta_r$

$$\beta'_r = \beta_r - y_{gr} \cdot \beta_e / y_{ge}$$

3. The constant vector $p_i^o$

$$\bar{p}_i^o = p_i^o - y_{ie} \cdot p_g^o / y_{ge} \qquad i \neq g$$

$$\bar{p}_g^o = p^o{}_g / y_{ge}$$

The basis variables will be

$$(x_1, x_2, \ldots, x_k)$$

They were

$$(x_1, x_2, \ldots, x_g, \ldots, x_k)$$

[notice the absence of $x_g$]

The constraints being exhausted would be

$$[b_1, b_2, \ldots, b_k]$$

they were

$$[b_1, b_2, \ldots, b_e, \ldots b_k] \qquad \ldots\ldots \text{(property 1)}$$

notice the absence of $b_e$

It is clear that the dimensions of the new matrix $A_{rr}^{-1}$ would be of order k-1.

In such a case, however, the original subvectors $a_{rg}$, and $a_{er}$ and the element $c_g$ must be re-read to be attached to the matrix;otherwise they would be neglected.

## 5.5 An emperical example

We will solve a more complicated example, where all the four previous cases being treated. The example is solved by a FORTRAN code written by the author The initial tableau of the problem is as follows (1) : The heading index $X_1, X_2, \ldots, X_{27}$ indicates the real variables of the problem. where the column index $X_{28}, X_{29}, \ldots, X_{57}$ are the index of the different constraints. The last row is, as usual, the objective function. When solving the example , the constraints will take the index $B_1, B_2, \ldots, B_{30}$ . The heading " S. MULT." means the simplex multipliers of the current basic variables. "B. VAR." means the current basic variables introduced into the solution. The column under this heading is the index of the basic variables. The parallel column index " B" indicates the constraints exhausted. The term " PROFITABILITY CHANCE OF THE REAL VARIABLES" means the simplex multipliers of the nonbasic variables.

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1) The example is taken from, Werner Zimmermann, Planungsrechnung, Friedr. Vieweg & Sohn, Braunschweig, 1968, pp. 132 - 133 - All the constraints have been considered to be of the form $AX \leq b$.

**Simplex - Tabelle 1**

| $X_*$ | $B^*$ | $X_1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X_{28}$ | 192 | 0,5 | 0,5 | 0,5 | 1 | 1 | 1 | | | | | | |
| 29 | 192 | | | | | | | 1,5 | 1,5 | 1,5 | 1,5 | 1,5 | 1,5 |
| 30 | 735 | 4,5 | 6,5 | 9 | 6,5 | 9 | 12 | 30 | 30 | 30 | 30 | | |
| 31 | 585 | | | | | | | | | | | 26 | 26 |
| 32 | 710 | 7 | 9 | 11 | 9 | 10 | 12 | 20 | 20 | | | 20 | 20 |
| 33 | 691 | | | | | | | | | 18 | 18 | | |
| 34 | 331 | 4,5 | 9 | 13,5 | 5,5 | 11 | 16,5 | | | | | | |
| 35 | 639 | | | | | | | 18 | 18 | 18 | 18 | 18 | 18 |
| 36 | 307 | 1,5 | 3 | 4,5 | 2 | 4 | 6 | 7,5 | 7,5 | 7,5 | 7,5 | 7,5 | 7,5 |
| 37 | 310,5 | 2,5 | 3,5 | 5 | 3 | 4 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 38 | 336 | 5 | 7 | 9 | 6 | 8 | 10,5 | 12 | | 12 | | 12 | |
| 39 | 307 | | | | | | | | 12 | | 12 | | 12 |
| 40 | 371 | 1,5 | 1,5 | 2,5 | 1,5 | 2 | 2,5 | 3 | 3 | 3 | 3 | 3 | 3 |
| 41 | 360 | 2,5 | 2,5 | 2,5 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 42 | 374 | 1 | 1,5 | 2 | 1 | 1,5 | 2 | 2,5 | 2,5 | 2,5 | 2,5 | 2,5 | 2,5 |
| 43 | 782 | 1 | 1 | 1,5 | 1 | 1,5 | 1,5 | 2 | 2 | 2 | 2 | 2 | 2 |
| 44 | 662 | 4,5 | 6 | 9 | 6 | 8 | 10 | 12 | 12 | 12 | 12 | 12 | 12 |
| 45 | 20 | 1 | | | | | | | | | | | |
| 46 | 16 | | 1 | | | | | | | | | | |
| 47 | 10 | | | 1 | | | | | | | | | |
| 48 | 14 | | | | 1 | | | | | | | | |
| 49 | 20 | | | | | 1 | | | | | | | |
| 50 | 10 | | | | | | 1 | | | | | | |
| 51 | 10 | | | | | | | 1 | 1 | 1 | 1 | 1 | 1 |
| 52 | 10 | | | | | | | | | | | | |
| 53 | 5 | | | | | | | | | | | | |
| 54 | 10 | | | | | | | | | | | | |
| 55 | 10 | | | | | | | | | | | | |
| 56 | 4 | | | | | | | | | | | | |
| 57 | 2 | | | | | | | | | | | | |
| $Z =$ | 0 | -110 | -190 | -210 | -220 | -190 | 180 | 580 | 440 | 490 | -340 | 470 | 330 |

| 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1,5 | 1,5 | 1,5 | 1,5 | 1,5 | 1,5 | 1,5 | 1,5 | 1,5 | 1,5 | 1,5 | 2 | 2 | 2 | 2 |
| | | 24 | 24 | 24 | 24 | | | | | | | | | |
| 26 | 26 | | | | | 20 | 20 | 20 | 20 | 36 | 30 | 24 | 50 | 55 |
| | | 22 | 22 | | | 22 | 22 | | | | | | | |
| 18 | 18 | | | 20 | 20 | | | 20 | 20 | 22 | 21 | 22 | 25 | 28 |
| 18 | 18 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 36 | 21 | 28 | 35 | 42 |
| 7,5 | 7,5 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 15 | 9 | 10 | 15 | 18 |
| 6 | 6 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 12 | 7,5 | 10 | 12,5 | 15 |
| 12 | | 15 | | 15 | | 15 | | 15 | | | | | | |
| | 12 | | 15 | | 15 | | 15 | | 15 | 18 | 15 | 18 | 21 | 24 |
| 3 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3,5 | 3 | 2,5 | 3,5 | 3,5 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 2,5 | 2,5 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 | 2,5 | 2 | 3,5 | 3,5 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 1,5 | 2 | 2 |
| 12 | 12 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 16 | 14 | 16 | 18 | 20 |
| | | | | | | | | | | | | | | |
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| | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | |
| 1 | 1 | | | | | | | | | | | | | |
| | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | |
| | | | | | | | | | | 1 | | | | |
| | | | | | | | | | | | 1 | | | |
| | | | | | | | | | | | | 1 | | |
| | | | | | | | | | | | | | 1 | |
| 170 | 230 | 430 | 260 | 320 | 150 | 360 | 190 | 250 | 70 | 470 | 450 | 380 | 500 | 600 |

THE OBJECTIVE FUNCTION IS 1200.00

PROFITABILITY CHANGE OF THE REAL VARIABLES

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X 2 | -190.00 | X 3 | -210.00 | X 4 | -220.00 | X 5 | -190.00 | X 6 | -180.00 | X 7 | -550.00 |
| X 9 | -440.00 | X10 | -430.00 | X11 | -540.00 | X12 | -470.00 | X13 | -530.00 | X14 | -370.00 |
| X15 | -230.00 | X16 | -430.00 | X17 | -260.00 | X18 | -320.00 | X19 | -180.00 | X20 | -360.00 |
| X21 | -190.00 | X22 | -250.00 | X23 | -70.00 | X24 | -470.00 | X25 | -450.00 | X26 | -380.00 |
| X26 | -500.00 | X 1 | -110.00 | X | | | | | | | |

REDUNDANT CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B 2 | 190.00 | B 3 | 735.00 | B 4 | 475.00 | B 5 | 710.00 | B 6 | 635.00 | B 7 | 331.00 |
| B 8 | 565.00 | B 9 | 271.00 | B10 | 240.00 | B11 | 336.00 | B12 | 259.00 | B13 | 264.00 |
| B14 | 354.00 | B15 | 367.00 | B16 | 778.00 | B17 | 622.00 | B18 | 21.00 | B19 | 16.00 |
| B20 | 10.00 | B21 | 14.00 | B22 | 20.00 | B23 | 10.00 | B24 | 10.00 | B25 | 10.00 |
| B26 | 5.00 | B27 | 10.00 | B28 | 10.00 | B29 | 4.00 | B 1 | 192.00 | B | |

2. ITERATION

| S.VOLT. | | B.VAR. | | INVERSION OF CURRENT SOLUTION | |
|---|---|---|---|---|---|
| 60.00 | B30 | Y27 | 2.00 | 1.00 | 0.00 |
| 580.00 | B24 | X 7 | 10.00 | 0.00 | 1.00 |

THE OBJECTIVE FUNCTION IS 7010.00

PROFITABILITY CHANGE OF THE REAL VARIABLES

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X 3 | -210.00 | X 4 | -220.00 | X 5 | -190.00 | X 6 | -180.00 | X 2 | -190.00 | X 8 | 140.00 |
| X 9 | 90.00 | X10 | 240.00 | X11 | 110.00 | X12 | 250.00 | X13 | 210.00 | X14 | 350.00 |
| X15 | -430.00 | X16 | -260.00 | X17 | -320.00 | X18 | -150.00 | X19 | -360.00 | X20 | -190.00 |
| X21 | -250.00 | X22 | -70.00 | X23 | -470.00 | X24 | -450.00 | X25 | -380.00 | X26 | -500.00 |
| X 1 | -110.00 | X | | | | | | | | | |

REDUNDANT CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B 3 | 435.00 | B 4 | 475.00 | B 5 | 510.00 | B 6 | 635.00 | B 7 | 331.00 | B 8 | 375.00 |
| B 9 | 196.00 | B10 | 220.50 | B11 | 216.00 | B12 | 259.00 | B13 | 334.00 | B14 | 324.00 |
| B15 | 342.00 | B16 | 758.00 | B17 | 502.00 | B18 | 21.00 | B19 | 16.00 | B20 | 10.00 |
| B21 | 14.00 | B22 | 20.00 | B23 | 10.00 | B 2 | 173.00 | B25 | 10.00 | B26 | 5.00 |
| B27 | 10.00 | B28 | 10.00 | B29 | 4.00 | B 1 | 192.00 | B | | | |

3. ITERATION

| S.VALT. | | B.VAR. | | INVERSION OF CURRENT SOLUTION | | |
|---|---|---|---|---|---|---|
| 600.00 | B30 | X27 | 2.00 | 1.00 | [illegible] | [illegible] |
| 580.00 | B24 | X 7 | 10.00 | 0.00 | 1.00 | 0.00 |
| 500.00 | B29 | X26 | 4.00 | 0.00 | 0.00 | 1.00 |

THE OBJECTIVE FUNCTION IS 9000.00

PROFITABILITY CHANGES OF THE REAL VARIABLES

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X 4 | -220.00 | X 5 | -190.00 | X 6 | -[illegible] | X 2 | -190.00 | X 8 | 14[illegible] | X 9 | 9[illegible] |
| X11 | 24[illegible] | X11 | 110.00 | X12 | [illegible] | X13 | 210.00 | X14 | 9[illegible] | X15 | -[illegible] |
| X16 | -260.00 | X17 | -320.00 | X18 | -150.00 | X19 | -360.00 | X20 | -19[illegible] | X21 | -25.00 |
| X22 | -70.00 | X23 | -470.00 | X24 | -45[illegible] | X25 | -38[illegible] | X 3 | -21[illegible] | X 1 | -11.00 |

REDUNDANT CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B 4 | 275.00 | B 5 | 51[illegible] | B 6 | 5[illegible] | B 7 | [illegible] | B 8 | 2[illegible] | B 9 | 130.00 |
| B10 | 170.00 | B11 | 216.00 | B12 | 17[illegible] | B13 | 32[illegible] | B14 | 3[illegible] | B15 | 320.00 |
| B16 | 750.00 | B17 | 430.00 | B18 | [illegible] | B19 | 16[illegible] | B21 | [illegible] | B21 | 14.00 |
| B22 | 20.00 | B23 | 10.00 | B 2 | 164.00 | B25 | 10[illegible] | B26 | [illegible] | B27 | [illegible] |
| B28 | 10.00 | B 3 | 435.00 | B 1 | 192.00 | B | | | | | |

4. ITERATION

| S.VALT. | | B.VAR. | | INVERSION OF CURRENT SOLUTION | | |
|---|---|---|---|---|---|---|
| 6[illegible] | [illegible] | X27 | 2.00 | [illegible] | [illegible] | [illegible] |
| 5[illegible] | B24 | X 7 | 10.00 | [illegible] | [illegible] | [illegible] |
| 5[illegible] | B29 | X26 | 4.00 | [illegible] | 1.00 | 0.00 |
| 47.00 | B28 | X23 | 5.00 | 1.00 | 0.00 | 1.00 |

THE OBJECTIVE FUNCTION IS 11[illegible]

PROFITABILITY CHANGES OF THE REAL VARIABLES

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X 5 | -19[illegible] | X 6 | -1[illegible] | X 2 | -1[illegible] | X 8 | 14[illegible] | X 9 | [illegible] | [illegible] | 2[illegible] |
| X11 | 11[illegible] | X12 | [illegible] | X13 | 21[illegible] | X14 | 3[illegible] | X15 | -3[illegible] | [illegible] | -[illegible] |
| X17 | -32[illegible] | X18 | -15[illegible] | X19 | -[illegible] | X2 | -19[illegible] | X21 | -2[illegible] | [illegible] | -[illegible] |
| X 4 | -22[illegible] | X24 | -45[illegible] | X25 | -3[illegible] | X 3 | -7[illegible] | X 1 | -1[illegible] | [illegible] | [illegible] |

REDUNDANT CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B 5 | 51[illegible] | B 6 | 42[illegible] | B 7 | [illegible] | B 8 | [illegible] | B 9 | 6[illegible] | B10 | 13[illegible] |
| B11 | 216.00 | B12 | 6[illegible] | B13 | 3[illegible] | B14 | 29[illegible] | B15 | 31[illegible] | B16 | [illegible] |
| B17 | 350.00 | B18 | 2[illegible] | B19 | 16[illegible] | B20 | [illegible] | B21 | [illegible] | B22 | 2[illegible] |
| B24 | 10.00 | B 2 | 157.00 | B25 | 1[illegible] | 4 | 99[illegible] | B27 | [illegible] | B28 | [illegible] |
| B 3 | 435.00 | B 1 | 192.00 | B | | | | | | | |

S.MULT. | B.VAR. | INVERSION OF CURRENT SOLUTION

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| -[illegible] | B30 | X27 | 2.[illegible] | 1.00 | [illegible] | [illegible] | [illegible] | [illegible] |
| 144.2[illegible] | B24 | X 7 | 10.00 | 0.00 | 1.00 | [illegible] | [illegible] | [illegible] |
| -249.99 | B29 | X26 | 4.00 | 0.00 | 0.00 | 1.00 | 0.00 | 0.00 |
| -401.42 | B26 | X23 | 5.00 | 0.00 | 0.00 | 0.00 | 1.00 | 0.00 |
| 21.42 | B 8 | X24 | 2.61 | -2.00 | -0.55 | -1.66 | -1.71 | 0.04 |

THE OBJECTIVE FUNCTION IS 12528.57

PROFITABILITY CHANGE OF THE REAL VARIABLES

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X 6 | -150.00 | X 2 | -190.00 | X 8 | 139.89 | X 9 | 89.99 | X10 | 239.99 | X11 | 109.99 |
| X12 | 349.99 | X13 | 209.99 | X14 | 349.99 | X15 | 64.28 | X16 | 254.28 | X17 | 164.28 |
| X18 | 364.28 | X19 | 154.28 | X20 | 324.28 | X21 | 264.28 | X22 | 444.28 | X 4 | -220.00 |
| X 5 | -190.00 | X25 | 219.99 | X 3 | -210.00 | X 1 | -110.00 | X | | | |

REDUNDANT CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B 6 | 370.00 | B 7 | 331.00 | B 5 | 510.00 | B 9 | 37.42 | B10 | 94.58 | B11 | [illegible] |
| B12 | 45.71 | B13 | 294.64 | B14 | 289.14 | B15 | 306.45 | B16 | 734.76 | B17 | [illegible] |
| B18 | 20.00 | B19 | 16.00 | B20 | 10.00 | B21 | 14.00 | B22 | 20.00 | B23 | [illegible] |
| B 2 | 152.76 | B25 | 10.00 | B 4 | 16.42 | B27 | 7.38 | B28 | 10.00 | B 3 | [illegible] |
| B 1 | 192.00 | B | | | | | | | | | |

6. ITERATION

S.MULT. | B.VAR. | INVERSION OF CURRENT SOLUTION

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| -202.31 | B30 | X27 | 2.00 | 1.00 | [illegible] | [illegible] | 0.00 | 0.00 |
| 656.65 | B24 | X 7 | 10.00 | 0.00 | 1.00 | 0.00 | 0.00 | 0.00 |
| -250.00 | B29 | X26 | 4.00 | 0.00 | 0.00 | 1.00 | 0.00 | 0.00 |
| 19.53 | B 4 | X23 | 5.93 | -0.32 | -1.66 | 0.00 | -0.06 | 0.09 |
| -6.48 | B 8 | X24 | 4.44 | -1.44 | 1.99 | -1.69 | 0.11 | -0.11 |

THE OBJECTIVE FUNCTION IS 12849.53

PROFITABILITY CHANGE OF THE REAL VARIABLES

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X 6 | -180.00 | X 2 | -180.00 | X 8 | 139.99 | X 9 | 89.99 | X10 | 239.99 | X11 | 617.96 |
| X12 | 757.96 | X13 | 717.96 | X14 | 557.96 | X15 | -585.55 | X16 | -415.55 | X17 | -475.55 |
| X18 | -305.55 | X19 | -124.81 | X20 | 45.18 | X21 | -14.81 | X22 | 165.18 | X 4 | -220.00 |
| X 5 | -190.00 | X25 | -92.59 | X 3 | -210.00 | X 1 | -110.00 | X | | | |

REDUNDANT CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B 6 | 355.09 | B 7 | 331.00 | B 5 | 510.00 | B 9 | 36.97 | B10 | 59.94 | B11 | 216.[illegible] |
| B12 | 37.50 | B13 | 292.89 | B14 | 260.66 | B15 | 305.28 | B16 | 733.24 | B17 | 304.[illegible] |
| B18 | 20.00 | B19 | 16.00 | B20 | 10.00 | B21 | 14.00 | B22 | 20.00 | B23 | [illegible] |
| B 2 | 150.20 | B25 | 10.00 | B26 | 1.06 | B27 | 5.55 | B28 | 10.00 | B 3 | 435.[illegible] |
| B 1 | 192.00 | B | | | | | | | | | |

7. ITERATION

| S.MULT. | | B.VAR. | | INVERSION OF CURRENT SOLUTION | | | | |
|---|---|---|---|---|---|---|---|---|
| -287.70 | B30 | X27 | 2.00 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 257.50 | B24 | X 7 | 10.00 | 0.00 | 1.00 | 0.00 | 0.00 | 0.00 |
| -240.99 | B29 | X26 | 4.00 | 0.00 | 0.00 | 1.00 | 0.00 | 0.00 |
| 2.45 | B 4 | X15 | 1.77 | -0.14 | -0.75 | 0.00 | -0.02 | 0.04 |
| 17.91 | B 8 | X24 | 9.16 | -1.83 | 0.00 | -1.56 | 0.03 | -0.03 |

THE OBJECTIVE FUNCTION IS 13886.45

PROFITABILITY CHANCE OF THE REAL VARIABLES

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X23 | 265.49 | X 2 | -190.00 | X 8 | 139.99 | X 9 | 89.99 | X10 | 239.95 | X11 | 273.93 |
| X12 | 313.91 | X13 | 273.91 | X14 | 413.91 | X 6 | -180.00 | X16 | 169.99 | X17 | 109.99 |
| X18 | 279.99 | X19 | 119.16 | X20 | 299.16 | X21 | 229.16 | X22 | 409.16 | X 4 | -220.00 |
| X 5 | -190.00 | X25 | 180.66 | X 3 | -210.00 | X 1 | -110.00 | X | | | |

REDUNDANT CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B 6 | 342.50 | B 7 | 331.00 | B 5 | 471.04 | B 9 | 39.33 | B10 | 57.56 | B11 | 189.43 |
| B12 | 37.50 | B13 | 255.99 | B14 | 27[illegible] | B15 | 300.54 | B16 | 72[illegible] | B17 | 2[illegible] |
| B18 | 20.00 | B19 | 16.00 | B20 | 16.00 | B21 | 24.00 | B22 | 20.00 | B23 | 10.00 |
| B 2 | 144.01 | B26 | 8.22 | B26 | 5.00 | B27 | 0.83 | B28 | 10.00 | B 3 | 392.50 |
| B 1 | 192.00 | B | | | | | | | | | |

8. ITERATION

| S.MULT. | | B.VAR. | | INVERSION OF CURRENT SOLUTION | | | | |
|---|---|---|---|---|---|---|---|---|
| 156.93 | B27 | X27 | 1.54 | -0.54 | 0.00 | -0.92 | 0.01 | [illegible] |
| 2[illegible] | B24 | X 7 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| 11.55 | B24 | X26 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| -2.77 | B 4 | X15 | 1.93 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| 17.91 | B 8 | X24 | 9.99 | 1.00 | [illegible] | [illegible] | [illegible] | [illegible] |

THE OBJECTIVE FUNCTION IS 14017.23

PROFITABILITY CHANCE OF THE REAL VARIABLES

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X23 | 75.18 | X 2 | -190.00 | X 8 | 139.99 | X 9 | 89.99 | X10 | 239.95 | X11 | [illegible] |
| X12 | 177.9[illegible] | X13 | 137.9[illegible] | X14 | 2[illegible] | X 6 | -1[illegible] | X16 | [illegible] | X17 | [illegible] |
| X18 | 27[illegible] | X19 | 14[illegible] | X20 | 1[illegible] | X21 | 1[illegible] | X22 | 30[illegible] | X 4 | -2[illegible] |
| X 5 | -190.00 | X25 | 55.12 | X 3 | -210.00 | X 1 | -110.00 | X | | | |

REDUNDANT CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B 6 | 337.72 | B 7 | 331.00 | B 5 | 469.58 | B 9 | 39.48 | B10 | 67.62 | B11 | [illegible] |
| B12 | [illegible] | B13 | 257.91 | B14 | 2[illegible] | B15 | 3[illegible] | B16 | [illegible] | B17 | [illegible] |
| B18 | 20.00 | B19 | 16.00 | B20 | 16.00 | B21 | 14.00 | B22 | 20.00 | B23 | [illegible] |
| B 2 | 143.15 | B25 | 8.15 | B26 | 5.00 | B27 | [illegible] | B28 | [illegible] | [illegible] | [illegible] |

| | | | | | INVERSION OF CURRENT SOLUTION | | | | |
|---|---|---|---|---|---|---|---|---|---|
| [illegible] | | | | | | | | | |
| 1 [illegible] | [illegible] | X27 | 1.84 | [illegible] | 0.00 | [illegible] | [illegible] | [illegible] | [illegible] |
| [illegible] | [illegible] | X 7 | 1[illegible] | 0.00 | 1.00 | [illegible] | [illegible] | [illegible] | [illegible] |
| [illegible] | [illegible] | X26 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| -2.77 | R 4 | X15 | 1.83 | 0.07 | -0.75 | 0.13 | [illegible] | [illegible] | [illegible] |
| [illegible] | R 8 | X24 | 9.99 | 1.00 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| [illegible] | [illegible] | X 4 | 14.00 | 0.00 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |

THE OBJECTIVE FUNCTION IS 17097.23

PROFITABILITY CHANGE OF THE REAL VARIABLES

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X 5 | -15.[illegible] | X 9 | 139.99 | X10 | [illegible]9.99 | X11 | 234.99 | X12 | 37.[illegible] | X13 | [illegible] |
| X14 | 137.50 | X14 | 277.99 | X 6 | -190.00 | X16 | 109.99 | X17 | [illegible] | X18 | [illegible] |
| X19 | 24.54 | X20 | 184.54 | X21 | 124.54 | X22 | 304.54 | X23 | -9.18 | X 8 | -29.[illegible] |
| X25 | [illegible] | X 3 | -210.00 | X 1 | -110.00 | A | | | | | |

REDUNDANT CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| [illegible]7 | 2[illegible] | R 5 | 343.58 | B 9 | 11.[illegible] | B10 | 46.62 | B11 | [illegible] | B12 | [illegible] |
| [illegible]8 | 2[illegible] | R14 | 235.85 | B15 | 2[illegible] | B16 | 719.23 | B17 | 1[illegible] | B18 | [illegible] |
| [illegible]4 | 16.[illegible] | R20 | 10.00 | B 6 | 3[illegible] | B22 | 2[illegible] | B23 | [illegible] | B 2 | [illegible] |
| [illegible]5 | [illegible] | R26 | [illegible] | B30 | 0.45 | B28 | 14.00 | B 3 | 299.92 | B 4 | [illegible] |

10. ITERATION

| S.VALT. | | B.VAR. | | | INVERSION OF CURRENT SOLUTION | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 145.41 | R27 | X27 | 1.54 | -2.54 | 0.00 | -[illegible] | 0.01 | -0.00 | [illegible] | [illegible] |
| 1[illegible] | R24 | X 7 | 1[illegible] | 0.00 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 28.49 | R29 | X26 | 4.00 | 0.00 | [illegible] | 1.00 | 0.00 | 0.00 | [illegible] | [illegible] |
| -6.16 | R 4 | X15 | 1.83 | 0.07 | -0.75 | 0.13 | -0.02 | 0.04 | [illegible] | [illegible] |
| 2.38 | R 8 | X24 | 8.99 | 1.00 | 0.00 | -[illegible] | 0.00 | +0.03 | [illegible] | [illegible] |
| 12[illegible] | R21 | X 4 | 14.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 45.56 | R 9 | X 3 | 2.55 | 0.04 | -0.33 | 0.06 | -0.01 | -0.07 | -0.44 | [illegible] |

THE OBJECTIVE FUNCTION IS 17633.19

PROFITABILITY CHANGE OF THE REAL VARIABLES

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X 5 | 134.99 | X 9 | 89.99 | X10 | 239.99 | X11 | -50.33 | X12 | 89.66 | X13 | [illegible] |
| X14 | 1[illegible] | X 6 | 99.99 | X16 | 170.00 | X17 | 110.00 | X18 | 280.00 | X19 | -53.33 |
| X20 | 110.16 | X21 | 56.66 | X22 | 236.66 | X23 | 93.00 | X 8 | -3.33 | X25 | [illegible] |
| X 2 | -50.00 | X 1 | -40.00 | X | | | | | | | |

REDUNDANT CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| R 5 | 315.80 | R 7 | 219.54 | B10 | 92.86 | B11 | 81.47 | B12 | 95.90 | B13 | 260.83 |
| R14 | 229.47 | R15 | 2[illegible] | B16 | 709.40 | B17 | 166.40 | B18 | 20.00 | B19 | 16.[illegible] |
| R20 | 7.44 | R 6 | 337.72 | B22 | 70.00 | B23 | 17.00 | B 2 | 143.15 | B25 | 5.16 |
| R26 | 5.00 | B30 | 0.45 | B28 | 10.00 | B 3 | 276.93 | B 1 | 176.72 | B | |

11. ITERATION

| S.[illegible] | | B.VAR. | | INVERSION OF CURRENT SOLUTION | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| -5.41 | B27 | X19 | 4.24 | -1.50 | [illegible] | -2.50 | 0.05 | -0.00 | [illegible] | [illegible] |
| 187.49 | B24 | X 7 | 1[illegible] | 0.00 | 1.00 | .00 | 0.00 | 0.00 | [illegible] | [illegible] |
| -1[illegible]7.63 | B29 | X26 | [illegible] | 0.00 | 0.00 | 1.00 | 0.00 | 0.00 | [illegible] | [illegible] |
| -3.5[illegible] | B 4 | X15 | 0.29 | 0.62 | -0.75 | 1.04 | -0.05 | 0.04 | [illegible] | [illegible] |
| 2.36 | B 8 | X24 | 9.99 | 1.00 | 0.00 | -1.00 | 0.00 | -0.00 | [illegible] | [illegible] |
| 126.[illegible]6 | B21 | X 4 | 1[illegible] | [illegible] | [illegible] | [illegible] | 0.00 | 0.00 | [illegible] | [illegible] |
| 46.66 | B 9 | X 3 | 3.92 | -0.44 | -0.33 | -0.74 | -0.0[illegible] | -0.07 | -0.44 | 0.22 |

THE OBJECTIVE FUNCTION IS 17959.85

PROFITABILITY CHANGE OF THE REAL VARIABLES

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X27 | 1[illegible] | X 9 | 5[illegible].9[illegible] | X[illegible] | 23[illegible] | X[illegible] | [illegible] | X12 | 156.[illegible]9 | X13 | [illegible] |
| X14 | 259.99 | X 6 | 99.99 | X1[illegible] | [illegible] | X1[illegible] | [illegible] | X18 | 280.[illegible] | X 5 | [illegible] |
| X2[illegible] | 17[illegible].00 | X21 | 110.00 | X22 | 29[illegible].00 | X23 | [illegible]6.5[illegible] | X 5 | -3.33 | X25 | [illegible] |
| X 2 | -[illegible] | X 1 | -40.00 | X | | | | | | | |

REDUNDANT CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B 5 | 240.99 | B 7 | 2[illegible].00 | B[illegible] | 27.53 | B[illegible] | [illegible] | B 2 | [illegible] | B[illegible] | 25[illegible] |
| B14 | 222.56 | B[illegible] | 279.[illegible] | B[illegible] | [illegible] | B[illegible] | [illegible] | B[illegible] | [illegible] | B[illegible] | [illegible] |
| B22 | 6.97 | B 6 | 38[illegible].99 | B[illegible] | [illegible] | B[illegible] | [illegible] | B 2 | [illegible] | B[illegible] | [illegible] |
| B26 | 5.00 | B3[illegible] | 2.00 | B28 | 10[illegible] | B 3 | 3[illegible]1.66 | B 1 | [illegible] | B | |

12. ITERATION

| S.[illegible] | | B.VAR. | | INVERSION OF CURRENT SOLUTION | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| [illegible] | B27 | X19 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| 1[illegible]9.59 | B2[illegible] | X[illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| [illegible]6 | B[illegible] | X26 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| [illegible] | B[illegible] | X 3 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| [illegible] | B 8 | X24 | [illegible]99 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| 1[illegible] | B21 | X 4 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |

THE OBJECTIVE FUNCTION IS 178[illegible].79

PROFITABILITY CHANGE OF THE REAL VARIABLES

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X1[illegible] | [illegible] | X27 | [illegible] | X 9 | [illegible] | X1[illegible] | 239.99 | X1[illegible] | -1[illegible]3.33 | X[illegible] | [illegible] |
| X1[illegible] | -1[illegible] | X14 | 12[illegible].66 | X 6 | [illegible] | X16 | 273.33 | X1[illegible] | [illegible]3.33 | X1[illegible] | [illegible] |
| [illegible] | 1[illegible] | X2[illegible] | 17[illegible] | X21 | [illegible] | X22 | 29[illegible].00 | X23 | 157.99 | [illegible] | [illegible] |
| X25 | [illegible] | X 2 | -5[illegible] | X 1 | -[illegible] | X | | | | | |

REDUNDANT CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B2[illegible] | [illegible] | B 5 | 229.63 | B 7 | [illegible] | B[illegible] | [illegible] | B[illegible] | [illegible] | B[illegible] | [illegible] |
| B[illegible] | 246.76 | B14 | 221.[illegible] | B15 | [illegible] | B1[illegible] | [illegible] | B1[illegible] | [illegible] | B[illegible] | [illegible] |
| B1[illegible] | 16.[illegible] | B20 | 5.65 | B 5 | 35[illegible] | B 2 | [illegible] | B23 | [illegible] | B 2 | [illegible] |
| B2[illegible] | [illegible] | B26 | [illegible] | B30 | [illegible] | B[illegible] | [illegible] | B 3 | 3[illegible].6[illegible] | B 1 | [illegible] |

21. ITERATION

| S.VALT. | | M.VAR. | | INVERSION OF CURRENT SOLUTION | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 [illegible].99 | R27 | X 2 | 10.99 | -0.00 | -0.00 | 0.00 | 0.00 | 1. | -0.00 |
| 1[illegible].79 | R24 | X 7 | 9.99 | 0.00 | 0.99 | 0.00 | -0.00 | -0.00 | -0.00 |
| 97.99 | R 9 | X25 | 5.99 | -0.89 | -0.38 | 0.09 | -0.02 | -0.08 | -0.02 |
| 1 .00 | R 1 | X 1 | 4.00 | -0.00 | -2.39 | -0.00 | 0.19 | -1.39 | -1.19 |
| 1.00 | R19 | X24 | 9.99 | 0.99 | -0.00 | -0.00 | 0.00 | 0.00 | -0.00 |
| [illegible].39 | R21 | X 4 | 14.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 1.00 |

THE OBJECTIVE FUNCTION IS 19139.99

PROFITABILITY CHANCE OF THE REAL VARIABLES

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X15 | 32.99 | X19 | 102.99 | X 9 | 89.99 | X10 | 112.79 | X11 | 109.99 | X12 | 122.79 |
| X13 | 209.99 | X14 | 222.79 | X 6 | 159.29 | X16 | 43.99 | X17 | 142.99 | X18 | 183.99 |
| X 5 | 12.79 | X20 | 113.99 | X21 | 212.99 | X22 | 233.99 | X23 | 99.99 | X 9 | 46.79 |
| X 3 | 56.39 | X27 | 83.99 | X26 | 69.99 | X | | | | | |

REDUNDANT CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B 4 | 140.99 | B 5 | 211.99 | B 7 | 91.99 | B 8 | 81.00 | B29 | 3.99 | B12 | 48.99 |
| B13 | 244.99 | B14 | 189.99 | B15 | 269.99 | B16 | 698.99 | B17 | 107.09 | B18 | 13.99 |
| B10 | 7.49 | B20 | 9.99 | B 6 | 348.99 | B22 | 20.00 | B23 | 10.00 | B 2 | 144.99 |
| B25 | 9.99 | B26 | 5.00 | B30 | 2.00 | B28 | 4.00 | B 3 | 221.99 | B 1 | 167.99 |

THE OPTIMAL SOLUTION AFTER 21 ITERATIONS

[illegible] FOR

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 15[illegible] | B27 | X19 | 6.[illegible] | -[illegible] | -[illegible].53 | -[illegible] | -[illegible] | [illegible] | -[illegible] | [illegible] |
| 1[illegible]3.52 | B24 | X 7 | 9.42 | -0.00 | 0.35 | [illegible].05 | -[illegible].03 | 0.[illegible]3 | [illegible].5 | -[illegible] |
| [illegible].19 | B 9 | X26 | 2.99 | -0.60 | -0.07 | [illegible].07 | [illegible] | -[illegible] | [illegible] | -[illegible]3 |
| -4.23 | B 4 | X 3 | 4.66 | 0.00 | -0.27 | 0.17 | -0.00 | -0.07 | -0.49 | 0.02 |
| 3.[illegible] | [illegible] | X2[illegible] | 9.99 | [illegible] | [illegible] | -[illegible] | -[illegible] | [illegible] | -[illegible] | [illegible] |
| 12[illegible].19 | B21 | X 4 | 14.[illegible] | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 1.00 | 0.00 |
| 9.23 | B11 | X11 | 0.57 | 0.00 | 0.64 | -0.05 | 0.03 | -0.03 | -0.05 | 0.07 |

THE OBJECTIVE FUNCTION IS 17956.52

PROFITABILITY CHANGE OF THE REAL VARIABLES

| | | | | | | |
|---|---|---|---|---|---|---|
| X2 [illegible] | [illegible] X 6 | 47.99 X10 | 201.[illegible] X15 | 14.61 X12 | 101.24 X13 | 99.9[illegible] |
| X1[illegible] | [illegible] | [illegible] | [illegible] | 124.[illegible] | 2[illegible] | [illegible] |
| X2[illegible] | [illegible] X21 | 1[illegible].99 X22 | 2[illegible]1.45 X23 | 92.33 X 5 | -3.53 X25 | 7.31 |
| X [illegible] | -[illegible] X 1 | -33.52 X | | | | |

[illegible] CAPACITY

| | | | | | | |
|---|---|---|---|---|---|---|
| B 5 | 2[illegible] B 7 | 191.[illegible] B1[illegible] | 24.66 B26 | [illegible] B12 | 9[illegible] B1[illegible] | [illegible] |
| B14 | 219.33 B19 | 27[illegible].16 B16 | 7[illegible] B17 | 137.[illegible] B18 | [illegible] | [illegible] |
| B2[illegible] | 5.33 B 6 | 40[illegible] B22 | [illegible] B23 | [illegible] B 2 | 1[illegible]1.99 [illegible] | [illegible] |
| B26 | 5.[illegible] B30 | 2.[illegible] B28 | 1.[illegible] B 3 | 319.30 B 1 | [illegible] | |

14. [illegible]

S[illegible]

INVERSION OF CURRENT SOLUTION

[illegible]

THE OBJECTIVE FUNCTION IS 1[illegible]1.59

PROFITABILITY [illegible]

[illegible] X21 ... [illegible] X22 ... 215.14 X23 ... 87.40 X 5 ... 56.67 X25

[illegible] X 1 ... -[illegible].51 X

[illegible] CAPACITY

[illegible] B 7 ... 193.[illegible] B19 ... [illegible] B29 ... [illegible] B12 ... [illegible] B13

[illegible] B16 ... 277.25 B16 ... 7[illegible]3.[illegible]7 B17

| SOLUTION | | R.VAR. | | INVERSION OF CURRENT SOLUTION | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1[illegible].35 | B27 | X19 | 4.50 | 0.21 | -0.58 | [illegible] | -0.[illegible] | [illegible] | [illegible] | [illegible] |
| 1[illegible]2.99 | B24 | X 7 | 7.14 | 0.45 | 0.41 | [illegible] | -[illegible] | [illegible] | [illegible] | [illegible] |
| [illegible].61 | B 9 | X25 | 5.03 | -0.93 | -0.14 | 0.12 | 0.00 | [illegible] | [illegible] | [illegible] |
| -4.23 | B 4 | X 2 | 9.21 | -0.45 | -0.46 | 0.30 | [illegible] | -0.20 | -0.62 | 0.01 |
| -1.94 | B 8 | X24 | 9.99 | 0.99 | -0.00 | -0.00 | -0.00 | 0.00 | -0.00 | 0.00 |
| 67.77 | B21 | X 4 | 14.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | [illegible] | 0.00 |
| 4.16 | B11 | X11 | 2.85 | -0.45 | 0.58 | [illegible] | 0.03 | -0.03 | [illegible] | 0.00 |

THE OBJECTIVE FUNCTION IS 18350.58

PROFITABILITY CHANCE OF THE REAL VARIABLES

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X2[illegible] | 24.41 | X 9 | 89.99 | X10 | 189.99 | X15 | 14.61 | X12 | 89.99 | X13 | 99.99 |
| X14 | 169.99 | X 6 | 185.41 | X16 | 122.11 | X17 | 124.61 | X18 | 232.11 | X 8 | 89.99 |
| X2[illegible] | 107.49 | X21 | 109.99 | X22 | 227.49 | X23 | 111.69 | X 5 | 57.[illegible] | X 3 | 68.[illegible] |
| X27 | 50.44 | X 1 | -8.74 | X | | | | | | | |

RED. UNUSED CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B 5 | 2[illegible]2.07 | B 7 | 171.07 | B10 | 14.89 | B29 | 3.99 | B12 | 66.35 | B13 | 254.5[illegible] |
| B14 | 206.35 | B15 | 277.10 | B16 | 702.23 | B17 | 119.14 | B18 | 20.00 | B19 | 6.76 |
| B[illegible] | 9.99 | B 6 | 370.21 | B22 | 20.00 | B23 | 10.00 | B 2 | 140.17 | B25 | 5.47 |
| B26 | 5.00 | B30 | 2.00 | B28 | 4.96 | B 3 | 369.65 | B 1 | 173.39 | B | |

16. ITERATION

| SOLUT. | | R.VAR. | | INVERSION OF CURRENT SOLUTION | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 127.5[illegible] | B27 | X19 | 2.86 | 0.29 | -0.49 | -0.19 | -0.00 | [illegible] | [illegible] | 0.05 |
| 1[illegible] | B24 | X 7 | 7.18 | 0.45 | 0.41 | 0.00 | -0.03 | [illegible] | -0.00 | -0.00 |
| [illegible].75 | B 9 | X25 | 6.43 | -1.00 | -0.21 | 0.16 | 0.00 | -0.02 | -0.03 | -0.05 |
| -4.23 | B 4 | X 1 | 17.80 | -0.88 | -0.90 | 0.59 | -0.00 | -0.21 | -1.31 | 0.02 |
| -[illegible] | B 8 | X24 | 9.99 | 0.99 | -0.00 | -0.00 | -0.00 | [illegible] | [illegible] | 0.00 |
| 76.24 | B21 | X 4 | 14.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | [illegible] | 0.00 |
| 4.36 | B11 | X11 | 2.81 | -0.45 | 0.59 | -0.00 | 0.03 | -0.03 | 0.00 | 0.00 |

THE OBJECTIVE FUNCTION IS 18506.38

PROFITABILITY CHANCE OF THE REAL VARIABLES

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X[illegible] | 1[illegible] | X 9 | 89.99 | X10 | 187.53 | X15 | 14.61 | X12 | 57.63 | X13 | 99.99 |
| X14 | 1[illegible].53 | X 6 | 218.54 | X16 | 119.15 | X17 | 124.61 | X18 | 229.15 | X 8 | 87.63 |
| X2[illegible] | 1[illegible].53 | X21 | 109.99 | X22 | 224.53 | X23 | 122.78 | X 5 | 50.[illegible] | X 3 | 93.21 |
| X27 | 65.97 | X26 | 37.35 | X | | | | | | | |

RED. UNUSED CAPACITY

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B 5 | 196.33 | B 7 | 173.87 | B10 | 1.69 | B29 | 3.99 | B12 | 41.12 | B13 | 241.46 |
| B1[illegible] | 1[illegible]5.57 | B15 | 273.58 | B16 | 694.80 | B17 | 94.77 | B18 | 2.19 | B19 | 13.97 |
| B2[illegible] | 9.99 | B 6 | 339.37 | B22 | 20.00 | B23 | 10.00 | B 2 | 139.82 | B25 | 7.13 |
| B26 | 5.[illegible] | B30 | 2.00 | B28 | 3.56 | B 3 | 348.34 | B 1 | 169.09 | B | |

17. ITERATION

| S.MULT. | | B.VAR. | | INVERSION OF CURRENT SOLUTION | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 77.53 | R27 | X19 | 9.96 | 0.29 | -0.40 | -0.19 | 0.05 | 0.07 | 0.03 |
| 219.15 | R24 | X 7 | 9.99 | 0.00 | 0.99 | 0.00 | -0.00 | 0.00 | -0.00 |
| 58.80 | R 9 | X25 | 6.43 | -1.00 | -0.21 | 0.16 | -0.05 | -0.02 | -0.03 |
| 4.50 | R11 | X 1 | 17.80 | -0.88 | -0.90 | 0.59 | 0.02 | -0.21 | -1.31 |
| -7.25 | R 8 | X24 | 9.99 | 0.99 | -0.00 | -0.00 | 0.00 | 0.00 | 0.00 |
| 76.33 | R21 | X 4 | 14.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 1.00 |

THE OBJECTIVE FUNCTION IS 18816.10

PROFITABILITY CHANGE OF THE REAL VARIABLES

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X11 | 109.99 | X 2 | 15.47 | X 9 | 89.89 | X10 | 185.91 | X15 | -70.00 | X12 | 195.91 |
| X13 | 209.99 | X14 | 295.91 | X 6 | 217.16 | X16 | 32.39 | X17 | 39.99 | X18 | 142.39 |
| X 8 | 85.91 | X20 | 102.39 | X21 | 109.99 | X22 | 222.39 | X23 | 143.52 | X 5 | 79.29 |
| X 3 | 92.95 | X27 | 144.92 | X26 | 120.77 | X | | | | | |

REDUNDANT CAPACITY

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| R 4 | 73.20 | R 5 | 196.33 | R 7 | 173.67 | R10 | 1.69 | R24 | 3.99 | R12 | 41.12 |
| R13 | 241.46 | R14 | 185.57 | R15 | 273.58 | R16 | 694.80 | R17 | 94.77 | R18 | 2.07 |
| R19 | 15.99 | R20 | 9.99 | R 6 | 349.87 | R22 | 2.00 | R23 | 10.00 | R 2 | 139.84 |
| R25 | 7.13 | R26 | 5.00 | R30 | 2.00 | R28 | 3.96 | R 3 | 263.87 | R 1 | 169.09 |

18. ITERATION

| S.MULT. | | B.VAR. | | INVERSION OF CURRENT SOLUTION | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 94.23 | R27 | X15 | 6.86 | 0.29 | -0.40 | -0.19 | 0.05 | 0.07 | 0.03 |
| 194.25 | R24 | X 7 | 9.99 | 0.00 | 0.99 | 0.00 | -0.00 | 0.00 | -0.00 |
| 44.50 | R 9 | X25 | 6.43 | -1.00 | -0.21 | 0.16 | -0.05 | -0.02 | -0.03 |
| 4.64 | R11 | X 1 | 17.80 | -0.88 | -0.90 | 0.59 | 0.02 | -0.21 | -1.31 |
| -7.32 | R 8 | X24 | 9.99 | 0.99 | -0.00 | -0.00 | 0.00 | 0.00 | 0.00 |
| 74.09 | R21 | X 4 | 14.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 1.00 |

THE OBJECTIVE FUNCTION IS 19016.60

PROFITABILITY CHANGE OF THE REAL VARIABLES

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X19 | 79.99 | X 2 | 44.86 | X 9 | 89.89 | X10 | 185.91 | X15 | 109.99 | X12 | 195.91 |
| X13 | 209.99 | X14 | 295.91 | X 6 | 217.16 | X16 | 32.39 | X17 | 39.99 | X18 | 142.39 |
| X 8 | 85.91 | X20 | 102.39 | X21 | 109.99 | X22 | 222.39 | X23 | 143.52 | X 5 | 79.29 |
| X 3 | 92.95 | X27 | 144.92 | X26 | 120.77 | X | | | | | |

REDUNDANT CAPACITY

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| R 4 | 130.50 | R 5 | 196.33 | R 7 | 173.67 | R10 | 1.69 | R24 | 3.99 | R12 | 41.12 |
| R13 | 241.46 | R14 | 185.57 | R15 | 273.58 | R16 | 694.80 | R17 | 94.77 | R18 | 2.07 |
| R19 | 15.99 | R20 | 9.99 | R 6 | 349.87 | R22 | 2.00 | R23 | 10.00 | R 2 | 139.84 |
| R25 | 7.13 | R26 | 5.00 | R30 | 2.00 | R28 | 3.96 | R 3 | 263.87 | R 1 | 169.09 |

| S.[illegible] | B.VAR. | | INVERSION OF CURRENT SOLUTION | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| [illegible] | B27 | X13 | [illegible] | -0.49 | -1.3[illegible] | [illegible] | .[illegible] | -0.33 | -[illegible] |
| 21[illegible] | B24 | X 7 | [illegible].99 | 0.00 | .99 | -[illegible] | -[illegible] | [illegible] | -[illegible] |
| 27.[illegible] | B 9 | X25 | [illegible].63 | -0.72 | 0.05 | -[illegible] | -[illegible]5 | [illegible] | [illegible] |
| [illegible].40 | B11 | X 1 | 19.49 | 1.49 | 1.50 | -[illegible].99 | -[illegible] | 0.99 | -[illegible] |
| 1[illegible].59 | B13 | X24 | 9.99 | 0.99 | -[illegible] | [illegible] | [illegible] | -[illegible] | -[illegible] |
| [illegible].59 | B21 | X 4 | 14.00 | 0.00 | 0.00 | [illegible] | [illegible] | [illegible] | 1.[illegible] |

THE OBJECTIVE FUNCTION IS 19[illegible]35.28

PROFITABILITY CHANGE OF THE REAL VARIABLES

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X10 | 7[illegible] | X 2 | -11.70 | X 9 | 89.99 | X1[illegible] | 139.19 | X11 | 1[illegible].99 | X1[illegible] | 1[illegible] | |
| X1[illegible] | 2[illegible] | X14 | 249.19 | X 6 | 13[illegible] | X16 | 44.00 | X17 | 11[illegible] | X1[illegible] | 13[illegible] | |
| X [illegible] | [illegible]9.1[illegible] | X2[illegible] | 114.00 | X21 | 1[illegible] | X22 | 234.00 | X23 | 67.[illegible] | X 5 | 2[illegible] | |
| X 3 | [illegible]2.1[illegible] | X27 | 51.00 | X26 | [illegible]2.5[illegible] | X | | | | | | |

REDUNDANT CAPACITY

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| [illegible] | 125.76 | B 5 | 196.89 | B 7 | 1[illegible].25 | B [illegible] | [illegible].01 | B29 | 3.9[illegible] | B12 | 3[illegible] | |
| B13 | 239.[illegible]6 | B14 | 1[illegible]2.[illegible] | B15 | 272.62 | B16 | 693.94 | B17 | 91.[illegible] | B18 | [illegible] | |
| B1[illegible] | 15.99 | B20 | 9.99 | B 6 | 3[illegible].03 | B22 | 2[illegible].00 | B23 | 1[illegible].00 | B 2 | 1[illegible] | |
| B25 | 7.[illegible]9 | B26 | 5.00 | B30 | 2.00 | B28 | 3.36 | B 3 | 201.04 | B 1 | 1[illegible] | |

20. ITERATION

| S.MULT. | B.VAR. | | INVERSION OF CURRENT SOLUTION | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1[illegible]4.99 | B27 | X 2 | 7.66 | -1.66 | -4.33 | 1.11 | [illegible].22 | -1.11 | -0.22 |
| 1[illegible] | B24 | X 7 | [illegible] | 0.00 | 1.99 | -[illegible] | -[illegible] | [illegible] | -[illegible] |
| [illegible] | B 9 | X25 | [illegible].74 | -0.74 | [illegible] | -[illegible] | -[illegible].05 | [illegible] | -[illegible] |
| 11.[illegible] | B11 | X 1 | 15.66 | 2.33 | 3.66 | -1.55 | -[illegible].11 | 1.55 | -[illegible] |
| -[illegible] | B1[illegible] | X24 | 9.99 | 0.99 | -[illegible] | [illegible] | [illegible].00 | -[illegible] | -[illegible] |
| 74.99 | B21 | X 4 | 14.0[illegible] | 0.0[illegible] | 0.[illegible] | [illegible] | 0.00 | [illegible] | 1.[illegible] |

THE OBJECTIVE FUNCTION IS 19124.98

PROFITABILITY CHANGE OF THE REAL VARIABLES

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X15 | 39.[illegible] | X19 | 10[illegible].00 | X 9 | 89.99 | X10 | 107.99 | X11 | 109.99 | X12 | 117.99 | |
| X13 | 2[illegible].99 | X14 | 217.99 | X 6 | 163.50 | X16 | 43.99 | X17 | 149.[illegible] | X18 | 153.99 | |
| X [illegible] | 7.99 | X20 | 113.99 | X21 | 219.00 | X22 | 233.99 | X23 | 105.[illegible] | X 5 | [illegible] | |
| X 3 | [illegible].99 | X27 | 59.99 | X26 | 74.99 | X | | | | | | |

REDUNDANT CAPACITY

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| B 4 | 123.[illegible] | B 5 | 205.33 | B 7 | 114.49 | B 8 | 6[illegible].00 | B29 | 3.99 | B12 | 35.[illegible] | |
| B13 | 238.12 | B14 | 179.41 | B15 | 269.33 | B16 | 694.54 | B17 | 33.49 | B18 | [illegible] | |
| B19 | 8.33 | B20 | 9.99 | B 6 | 332.50 | B22 | 20.00 | B23 | 1[illegible].00 | B 2 | 14[illegible] | |
| B25 | 9.99 | B26 | 5.00 | B30 | 2.00 | B28 | 3.25 | B 3 | 223.[illegible] | B 1 | 16[illegible].32 | |

DISCUSSION:

Examining the solved example(pp. 160-172), it might be clear that at the first iteration only one element needs to be inverted, namely a(30,27) . The index (27) indicates that the variable $X_{27}$ , having the most negative cost coefficient, namely (-600), has been chosen to enter the solution. The index (30) indicates that the constraint $B_{30}$ has been chosen to be replaced by the variable $X_{27}$ . The element a(30,27) has been defined according to the usual rule:

$P^{o}_{30} / a_{30,27}$ is the min. nonneg. of

the ratio $P^{o}_{i} / a_{i,27}$ $i=1,2,\ldots,30$

To move towards an adjacent extreme point, the variable having the most negative cost coefficient must enter the solution. Examining the "Profitability chance of the nonbasic variables" , the variable $X_7$ , having the most negative value, namely (-550), would be chosen to improve the objective function.

At the second iteration, only four elements need to be inverted, namely:

$a_{30,27}$ $a_{30,7}$

$a_{24,27}$ $a_{24,7}$

It must be clear that the main difference between the KM and the MKM is that; applying the KM the inverse of the succesive bases has to be computed at every iteration from the original system. Applying the MKM , the inverse of the current solution is derived directly from the preceding one as suggested by the Kairo Algorithm.

The question now is: is it possible that the submatrix $A_{rr}$ , as previously defined, could be at all a singular one ? The answer is definetly NO . To explain this point let us assume that,at a certain iteration, the submatrix $A_{rr}$ ; the basis of the current solution, is nonsingular. This can be done without loss of generality, since we must have, at least at the first iteration, a nonsingular submatrix. (§) Now assume that the variable $x_e$ is chosen to enter the solution. In such a case, the vector $Y_{ie}$ $i=1,2,\ldots,m$ , which is the value of the newcoming variable $x_e$ in terms of the current basis, is defined through:

$$Y_{re} = A_{rr}^{-1} \; a_{re} \qquad r=1,2,\ldots,k.$$

$$Y_{de} = a_{de} - A_{dr} \cdot Y_{re} \qquad d=k+1,k+2,\ldots,m.$$

where r+d=m

To define the pivot row, we determine as usual the min. nonneg. of the ratio:

$$P_i^o \; / \; Y_{ie} \quad \text{let it be} \quad P_g^o \; / \; Y_{ge}$$

Asserting that the ratio $P_g^o \; / \; Y_{ge}$ is the minimum nonnegative of $P_i^o \; / Y_{ie}$ does mean that the element $Y_{ge}$ must be nonzero , otherwise it would not have been chosen as a denominator. Having the element $Y_{ge}$ as a nonzero value means that the inverse of the enlarged matrix could be always defined.

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(§) At the first iteration only one element needs to be inverted.

## 5.6 The modified Kairo Method and the Problems of Post - optimization

Given an optimum solution for a linear programming problem, there may arise the need to study the effect of changes in the given values of $c_j$, $a_{ij}$, and $P_i$. Such a need arises when management wishes to evaluate different optimum solutions for different assumed values of $c_j$, $a_{ij}$, and $P_i$. (1)

This must be of interest since the given information are nothing more than estimations. Moreover, such study provides management with helpful information regarding evaluating the effectiveness of different policies. For instance, management, trying to maximize profits, may increase the supply of a certain capacity,or may try to more efficiently utilize an existing facility.

On the other hand, there may arise the need to evaluate the effect of considering a new variable or adding an extra constraint to a problem for which an optimum solution has already been found.

In the following section, we will show how we can evaluate the effect of such changes with the minimum computational effort compared with the efforts required when the Simplex method were in use.

Since the post-optimization study deals, basically, with the inverse of the basis, there must be, a proiri, an acceptance that using the MKM will simplify the study since the dimensions of the inverse of the basis $A_{rr}^{-1}$ given by the MKM would be lower than $B^{-1}$,

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(1) CHAIHO KIM, Intr. to Linear Programming.
Holt, Reinhart and Winston, Inc. 1971, New York, p. 291

Dinkelbach, W., Sensitivitätsanalysen und parametrische Programmierung,Berlin - Heidelberg, New York , Springer, 1969.

Noltemeier,H., Sensitivitätsanalyse bei diskreten linearen Optimierungsproblemen, Berlin-Heidelberg,Springer 1970

In certain cases, using the new method will even enable us to get immediate answeres regarding the effects of certain changes which is not possible if the simplex method or one of its variants were used. Theses premises are, however, the subject matter of the present chapter.

In the following analysis, we will assume that the optimal solution for the LP problem

$$\begin{array}{ll} \text{Max} & C^{t} x \\ \text{Subject to} & A\, x \lesseqgtr p \\ \text{and} & x \geqslant o \end{array}$$

could be put in the following form

| | $b_i \ldots b_r \ldots b_k$ | $x_{k+1} \cdot \cdot \cdot x_n$ | |
|---|---|---|---|
| $x_1$ <br> $\vdots$ <br> $x_r$ <br> $\vdots$ <br> $x_k$ | $A_{rr}^{-1}$ <br> $(Y_{rr})$ | $A_{rs}$ | $\bar{P}_r^o$ |
| $b_{k+1}$ <br> $\vdots$ <br> $b_m$ | $A_{dr}$ | $A_{ds}$ | $\bar{P}_d^o$ |
| | $\beta_r$ | $\beta_s$ | |

where the first K real variables have replaced the first K constraints and

$$\bar{P}_r = A_{rr}^{-1} \cdot P_r$$

$$\bar{P}_d^{\circ} = P_d - A_{dr} \cdot \bar{P}_r$$

$$\beta_r = c_r A_{rr}^{-1}$$

$$\beta_s = \beta_r A_{rs} - c_s$$

5.6.1 A change in C :

It might be clear that a change in a coefficient of the objective function CX will have no effect on the feasibility of the optimum solution.

Since C is not a part of:

$$\bar{P}_r = A_{rr}^{-1} \cdot P_r$$

$$\bar{P}_d = P_d - A_{d_r} \cdot \bar{P}_r$$

whether the given optimum solution will still be optimum or not if $C_f$ to become $\bar{C}_f$ may be evaluated in the following manner.

First, we have to find out whether the variable $x_f$ is a basic or a nonbasic variable. In other words, whether f is an element of r or of S $\quad r = 1,2,\ldots,k. \quad S = k+1,k+2,\ldots,n$ .

A change in $C_S$

Assuming that a price of a non-basic variable $C_f$ is to become $\bar{C}_f$. The effect of such a change could be evaluated using the formula

$$\bar{\beta}_f = z_f - c_f = \beta_r a_{rf} - \bar{c}_f$$

For a maximizing problem if $\bar{\beta}_f$ is still positive, the solution in question would still be optimal in spite of the price change. Generally, in a maximization problem, there will be no effect on the optimality if $\bar{c}_f < c_f$ . Now suppose that $\bar{c}_f > c_f$ . In such a case, there arises a possibility that the optimality condition may be affected. It must be apparent that while a very small increase in $c_f$ may still not effect the optimality condition, a very large change in $c_f$ will probably have an affect. The extent to which $c_f$ can change without affecting the optimality condition can be specified by

$$\lim \Delta c_f = \beta_f \qquad (1)$$

If $\Delta c_f > \beta_f$, the optimality condition will be violated.

A change in $C_r$ . the price of a basic real variable $X_r$ .

Suppose that $C_b$ is to be changed into $\bar{C}_b$, the new optimality evaluator may be evaluated by

$$\bar{\beta}_r = \sum_{\substack{r=1 \\ r\neq b}}^{k} C_r Y_{rr} + \bar{C}_b Y_{br}$$

which can be modified to

$$\bar{\beta}_r = \sum_{r=1}^{k} C_r Y_{rr} - C_b Y_{br} + \bar{C}_b Y_{br}$$

$$\bar{\beta}_r = \sum_{r=1}^{k} \beta_r + Y_{br} \Delta C_b$$

$\beta_s$ could be evaluated through:

since $\beta_s = \beta_r A_{rs} - C_s$

then $\bar{\beta}_s = \bar{\beta}_r A_{rs} - C_s$

$$\bar{\beta}_s = (\beta_r + Y_{br} \Delta C_b) A_{rs} - C_s$$

$$\bar{\beta}_s = \beta_r A_{rs} - C_s + Y_{br} A_{rs} \Delta C_b$$

$$\bar{\beta}_s = \beta_s + Y_{br} A_{rs} \Delta C_b$$

$$\bar{\beta}_s = \beta_s + Y_{bs} \Delta C_b \qquad \text{since } Y_{bs} = Y_{br} A_{rs} \qquad (\S)$$

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(1) CHAIHO KIM, op cit. p. 293

(2) Examine Page (47). This step must be similar to what should be done if the revised simplex method or the product form of inverse were used. In such a case we multiply $Y_{bj}$, which is the b+h vector in $B^{-1}$, by the nonbasic vectors $A_{js}$ vector by vector. Using the MKM, however, will make it easier, since we multiply the subvector $Y_{br}$ by the subvectors $A_{rs}$, where it is assumed that K/m

The optimality condition then requires that

$$\bar{\beta}_j = \beta_j + Y_{bj} \Delta C_b \geq 0 \qquad j = 1,2,\ldots,n.$$

This means that the optimal solution will no longer be optimal if

$$\beta_j + Y_{bj} \Delta C_b < 0$$

Since $\beta_j \geq 0$ for all $j$, $j = 1,2,\ldots,n$ in an optimal solution, then $\bar{\beta}_j$ cannot be negative so long as

$$Y_{bj} \Delta C_b \geq 0$$

Suppose that $Y_{bj} = 0$. Then $Y_{bj}(\bar{c}_b - c_b) = 0$ and $\bar{\beta}_j = \beta_j$ regardless of the value of $\bar{c}_b$.

Suppose now that $Y_{bj} > 0$. Then $Y_{bj}(\bar{c}_b - c_b) \geq 0$ so long as $\bar{c}_b > c_b$.

In such a case, a possibility that the optimality condition may be affected by the price occurs only when $\bar{c}_b < c_b$.

Generally, so long as the inequality

$$\bar{c}_b - c_b \leq \frac{\beta_j}{-y_{bj}} \quad , \quad \Delta c_b \leq \frac{\beta_j}{-y_{bj}}$$

holds, the optimal solution will not be violated.

It is possible to establish the upper and lowel limits for $\Delta c_b$ so that if $\Delta c_b$ is within these limits, the optimality conditions would not be affected

upper limits $\Delta c_b = \min \dfrac{\beta_j}{-y_{bj}}$ for all $y_{bj} < 0$

lower limits $\Delta c_b = \max \dfrac{\beta_j}{-y_{bj}}$ for all $y_{bj} > 0$

## 6. 2 A change in the right hand side $P_i$

Using the MKM, the optimal solution would be put in the following form

| | | | |
|---|---|---|---|
| $X_r$ | $A^{-1}_{(Y_{rr})}$ | $A_{rs}$ | $\bar{P}_r$ |
| $b_d$ | $A_{dr}$ | $A_{ds}$ | $\bar{P}_d$ |
| | $\beta_r$ | $\beta_s$ | |

where $\bar{p}_r = A^{-1}_{rr} \cdot p_r$ is the value of the optimal solution $X_r$, and

$\sum_{k+1}^{m} \bar{P}_d = P_d - A_{dr}\bar{P}_r$ are nothing more than redundant capacities.

It must be clear that a change in $P_i$ will not affect the optimality conditions. Moreover, an increase in $P_d$ will lead only to an increase in $P_d$, the redundant capacity. While a very small decrease in $P_d$ may still not affect the feasibility condition, a very change in $P_d$ will probably have an affect.

The extent to which $P_d$ can decrease without affecting the feasibility condition can be specified by:

$$\text{Lim} \; - \Delta P_d = \bar{P}_d$$

If $-\Delta P_d > \bar{P}_d$, the feasibility condition will be violated.

On the other hand a change in $P_r$ may affect the feasibility since

$$\bar{P}_r = A^{-1}_{rr} P_r$$

$$\bar{P}_d = P_d - A_{dr} A^{-1}_{rr} P_r$$

$$\bar{P}_d = P_d - A_{dr} \bar{P}_r$$

Assuming that $P_b$ is to be changed to $\hat{P}_b$ where the index b is an element of $r, r = 1,2,\ldots,k$.

Then

$$\bar{\bar{P}}_r = \sum_{\substack{r=1 \\ r \neq b}}^{k} Y_{rr} P_r + y_{rb} \hat{P}_b$$

$$\bar{\bar{P}}_r = \sum_{r=1}^{n} Y_{rr} P_r + y_{rb} \hat{P}_b - y_{rb} P_b$$

$$\bar{\bar{P}}_r = \bar{P}_r + y_{rb} (\hat{P}_b - P_b)$$

$$\bar{\bar{P}}_r = \bar{P}_r + y_{rb} \cdot \Delta P_b$$

$\bar{\bar{P}}_d$ could be defined through:

$$\bar{\bar{P}}_d = P_d - A_{dr} (\bar{\bar{P}}_r)$$

$$\bar{\bar{P}}_d = P_d - A_{dr} \bar{P}_r - A_{dr} y_{rb} \cdot \Delta P_b$$

$$\bar{\bar{P}} = \bar{P}_d - A_{dr} y_{rb} \Delta P_b$$

$$= \bar{P}_d + y_{db} \Delta P_b \quad \text{since} \quad y_{db} = -A_{dr} y_{rb}$$

whichs means that

$$\bar{\bar{P}}_i = \bar{P}_i + y_{ib} \cdot \Delta P_b \geq 0$$

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(8) Examine Page (44) - It might be clear that there is no need to perform such a step when the revised simplex were used. $y_{db}$ is a part of $B^{-1}$

The last equation shows that so long as $y_{ib} \Delta p_b \geq 0$, $\bar{\bar{P}}_i$ will not become negative since $\bar{P}_i \geq 0$. If $y_{ib} \cdot \Delta p_b < 0$, $\bar{\bar{P}}_i$ might become negative.

Since $\bar{\bar{P}}_i$ must be nonnegative, we must have:

$$\bar{P}_i \geq - y_{ib}(\hat{p}_b - p_b)$$

$$\hat{p}_b - p_b \geq \frac{\bar{P}_i}{-y_{ib}}$$

$$p_b \geq \frac{\bar{P}}{-y_{ib}}$$

if the given solution is to remain feasible. The upper and lower limits for $\Delta p_b$ so that the feasibility condition would not be violated, could be specified as follows

$$\text{upper limits } \Delta p_b = \min \frac{\bar{P}_i}{-y_{ib}} \quad \text{for all } y_{rb} < 0$$

$$\text{lower limits } \Delta p_b = \max \frac{\bar{P}_i}{-y_{ib}} \quad \text{for all } y_{rb} > 0$$

A change in an Input - Output coefficient $a_{ij}$ :

considering that the last - optimal tableau is as follows,

| | | |
|---|---|---|
| $A^{-1}_{rr}{}_{(Y_{rr})}$ | $A_{rs}$ | $\bar{P}_r$ |
| $A_{dr}$ | $A_{ds}$ | $\bar{P}_d$ |
| $\beta_r$ | $\beta_s$ | |

it might be easy to conclude that a change in $A_{ds}$ d=k+1, k+2,...,m s=k+1, k+2,...,n will have no affect on both of the optimality and feasibility conditions.(6)

This is a fact since $A_{ds}$ is not a part of

$$\bar{P}_r = Y_{rr} \cdot P_r$$

$$\bar{P}_d = P_d - A_{dr} \bar{P}_r \qquad \text{feasibility condition}$$

and

$$\beta_r = C_r Y_{rr}$$

$$\beta_s = \beta_r A_{rs} - C_s \qquad \text{optimal conditions}$$

A change in $A_{rs}$, $A_{ds}$ and $A_{rr}$ will affect, on the contrary, both of the feasibility and optimality conditions.

A change in $A_{rs}$ :

A change in $A_{rs}$ will have no effect on the feasibility condition since $A_{rs}$ is not a part of

$$\bar{P}_r = Y_{rr} \cdot P_r$$

$$\bar{P}_s = P_s - A_{dr} \bar{P}_r$$

However, a change in $A_{rs}$ will effect the optimality condition since $A_{rs}$ is a part of

$$\beta_s = \beta_r A_{rs} - C_s$$

(6)Applying the simplex method or one of its variants, we are not able to assert such a conclusion.

Let the coefficient $a_{bt}$ is to become $\bar{a}_{bt}$ where t is an element of s, s=k+1,k+2,...,n and b is an element of r, r=1,2,...,k then

$$\bar{\beta}_t = \sum_{r=1}^{k} \beta_r a_{rt} - \beta_b a_{bt} + \beta_b \bar{a}_{bt} - c_t$$

$$\bar{\beta}_t = \beta_r a_{rt} - c_t + \beta_b (\bar{a}_{bt} - a_{bt})$$

$$= \beta_t + \beta_b (\bar{a}_{bt} - a_{bt})$$

To maintain the optimality of the solution we must have

$$\bar{\beta}_t = \beta_t + \beta_b (\bar{a}_{bt} - a_{bt}) \geq 0$$

$$\beta_t \geq \beta_b (a_{bt} - \bar{a}_{bt})$$

$$a_{bt} - \bar{a}_{bt} \leq \frac{\beta_t}{\beta_b}$$

Since the right-hand side is a positive value (optimality condition), then the optimality condition will not be affected so long as $\bar{a}_{bt} > a_{bt}$. While a small decrease in $a_{bt}$ might not affect the optimal solution, a great decrease might make it profitable to introduce $x_t$ in the solution when

$$\frac{\beta_t}{\beta_b} = a_{bt} - \bar{a}_{bt} \quad \text{where } a_{bt} > \bar{a}_{bt}$$

A change in $A_{dr}$

A change in $A_{dr}$ $(d=k+1,k+2,\ldots,m.$ $r=1,2,\ldots,k)$ may affect only the feasibility condition since,

$$\sum_{d=k+1}^{m} \sum_{r=1}^{k} \bar{P}_d = P_d - A_{dr} \cdot \bar{P}_r$$

Suppose that $a_{tb}$ is to become $\bar{a}_{tb}$ where t and b are elements of d and r respectively.

$$\bar{\bar{P}}_t = P_t - A_{tr}\bar{P}_r + a_{tb} \cdot \bar{P}_b - \bar{a}_{tb} \cdot \bar{P}_b$$

$$\bar{\bar{P}}_t = P_t - A_{tr}\bar{P}_r - \bar{P}_b(\bar{a}_{tb} - a_{tb})$$

$$\bar{\bar{P}}_t = \bar{P}_t - \bar{P}_b\ (\bar{a}_{tb} - a_{tb}) \quad \geq \quad 0$$

Since $\bar{\bar{P}}_t$ must be nonnegative, then we must have

$$\bar{P}_t \geq \bar{P}_b(\bar{a}_{tb} - a_{tb})$$

$$\Delta\ a_{tb} \leq \frac{\bar{P}_t}{\bar{P}_b}$$

Since the right-hand side of the last inequality must be non-negative, any decrease in $a_{tb}$ will not violate the feasibility condition. An increase may, on the contrary, affect the feasibility condition. The extent to which $a_{tb}$ can increase without affecting the feasibility condition is

$$\lim \Delta\ a_{tb} = \frac{\bar{P}_t}{\bar{P}_b}$$

A change in $A_{rr}$

Regardless of the reasons that cause a change in $a_{bf}$ where f and b is an element of r,r = 1,2,...,k, it must be clear that a change in $a_{bf}$ may affect both of the feasibility and the optimality conditions, since $a_{bf}$ is a part of $A_{rr}$

$$\bar{P}_r = A_{rr}^{-1} \cdot P_r \qquad \text{[feasibility condition]}$$

and

$$\beta_r = C_r \cdot A_{rr}^{-1} \qquad \text{[optimality condition ]}$$

Indeed, if $a_{bf}$ is to become $\bar{a}_{bf}$, we may consider the case as if a new product should be introduced in place of a product that is currently in the solution having the same coefficients $a_{rf}$ $r \neq b$, $r=1,2,...,k$) and $\bar{a}_{bf}$.

Since

$$A_{rr}^{-1} \cdot A_{rr} = I$$

and

$$A_{rr}^{-1} \cdot a_{rf} = \begin{matrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{matrix} \qquad \text{the fth. position}$$

then

$$\sum_{r=1}^{k} y_{rr} \cdot \begin{pmatrix} a_{1f} \\ \vdots \\ a_{bf}+\Delta a_{bf} \\ \vdots \\ a_{kf} \end{pmatrix} = 1 + y_{fb} \cdot \Delta a_{bf} = \hat{y}_{rf} \qquad \begin{pmatrix} y_{1b} \cdot \Delta a_{bf} \\ \vdots \\ y_{kb} \cdot \Delta a_{bf} \end{pmatrix}$$

$\sum_{d=k+1}^{m} \hat{y}_{df}$ could be defined through

$$\hat{y}_{df} = a_{df} - A_{dr}\,\hat{y}_{rf}$$

$$= a_{df} - (a_{d1} \cdot y_{1b} \cdot \Delta a_{bf}, \ldots, a_{df} + a_{df} y_{fb}, \ldots, a_{dk}\, y_{kb} \cdot \Delta a_{bf}) =$$

$$= - \Delta a_{bf} A_{dr} y_{rb} \qquad \begin{matrix} d=k+1,k+2,\ldots,m \\ r=1,2,\ldots,k \end{matrix}$$

$$= y_{db} \cdot \Delta a_{bf} \qquad \text{since } y_{db} = - A_{dr} \cdot y_{rb}$$ (§)

since $\hat{x}_f$ must replace $x_f$, we must pivot on $\hat{y}_{ff}$. In order to maintain the feasibility we must have:

(1) $\hat{y}_{ff} = 1 + y_{fb} \cdot \Delta a_{bf} > 0$

and

(2) $\sum_{i=1}^{m} \bar{P}_i - \dfrac{\bar{P}_f \cdot y_{ib}\ \Delta a_{bf}}{1 + y_{fb} \cdot \Delta a_{bf}} \geq 0$

Then the upper and lower boundaries for $\Delta a_{bf}$ are (1)

$$\text{upper limits } \Delta a_{bf} = \text{Min } i \neq f \quad \frac{\bar{P}_i}{y_{ib}\bar{P}_f - y_{fb} \cdot \bar{P}_i}$$

$$\text{for } y_{ib} > 0 \text{ and } a_{ib}\bar{P}_f - y_{fb}\bar{P}_i > 0$$

$$\text{lower limits } \Delta a_{bf} = \text{Max } i \neq f \quad \frac{\bar{P}_i}{y_{ib}\bar{P}_f - y_{fb} \cdot \bar{P}_i}$$

$$\text{for } y_{ib} < 0 \text{ and } y_{ib}\,\bar{P}_f - y_{fb} \cdot \bar{P}_i < 0$$

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(§) Examine Page (44)
(1) CHAIHO KIM, op cit, p. 308

These results could obtained as well if we consider that the old basic variable $x_f$ must be replaced with the new variable $\bar{x}_f$ whose alue in terms of the current basis is

$$\begin{vmatrix} y_{1b} \Delta a_{bf} \\ y_{2b} \Delta a_{bf} \\ \vdots \\ 1 + y_{fb} \Delta a_{bf} \\ \vdots \\ y_{ib} \cdot \Delta a_{bf} \end{vmatrix} \longrightarrow \text{(pivot element)}$$

To maintain the feasibility condition, we must have

First: the ratio $\bar{p}_f$ / $1 + y_{fb} \Delta a_{bf}$ must be nonnegative. Since $\bar{p}_f$ is nonnegative (feasibility condition) then $(1+a_{fb} \Delta a_{bf})$ must be nonnegative

Second:

$$\frac{\bar{p}_f}{1 + y_{fb} \Delta a_{bf}} \leq \frac{\sum_{i=1}^{m} \bar{p}_i}{\sum_{i=1}^{m} y_{ib} \cdot \Delta a_{bf}}$$

Let us now evaluate the affect of a change in $a_{bf}$ on the optimality.

Recalling that

$$\beta_f = \beta_r \cdot a_{rf} - c_f = 0$$

then

$$\hat{\beta}_f = \beta_r \cdot a_{rf} - \beta_b \cdot a_{bf} + \beta_b \hat{a}_{bf} - c_f$$

$$= \beta_r a_{rf} - c_f + \beta_b (\hat{a}_{bf} - a_{bf})$$

$$\hat{\beta}_f = \beta_b (\hat{a}_{bf} - a_{bf}) \qquad \text{since } \beta_r \cdot a_{rf} - c_f = 0$$

$$\hat{\beta}_f = \beta_b \Delta a_{bf}$$

It might be clear that when $\beta_f < 0$, the objective function will increase with

$$\hat{\beta}_f \cdot \frac{\bar{p}_f}{1 + y_{fb} \cdot \Delta a_{bf}}$$

if $\hat{\beta}_f > 0$, the objective function will decrease with the same value. Since $\beta_b \geq 0$, then any increase in $a_{bf}$ will not be profitable.

To maintain the optimality we must have

$$\hat{\beta}_r = \beta_r - \frac{(\beta_b \Delta a_{bf}) \cdot y_{fr}}{1 + y_{fb} \cdot \Delta a_{bf}} \geq 0$$

This implies that

$$\beta_r \geq \frac{\beta_b \Delta a_{bf} \cdot y_{fr}}{1 + y_{fb} \cdot \Delta a_{bf}}$$

which could be modified into

---

(§) Notice that the formula is the product of the usual transformation procedure.

(%) $\beta_b \cdot \Delta a_{bf} = \beta_f$ is the simplex multiplayer [illegible] any new variable $y_f$

$$\text{upper limits } \Delta a_{bf} = \min \quad \frac{\beta_r}{\beta_b y_{fr} - \beta_r y_{fb}}$$

$$\text{for } y_{fr} > 0 \text{ and } \beta_b y_{fr} - \beta_r y_{fb} > 0$$

$$\text{lower limits } \Delta a_{bf} = \max \quad \frac{\beta_r}{\beta_b y_{fr} - \beta_r y_{fb}}$$

$$\text{for } y_{fr} < 0 \text{ and } \beta_b y_{fr} - \beta_r y_{fb} < 0$$

As indicated before that when

$$\hat{\beta}_f = \beta_b \Delta a_{bf} < 0$$

the objective function would increase with

$$\frac{\beta_b \Delta a_{bf} \cdot \bar{p}_f}{1+y_{fb} \cdot \Delta a_{bf}}$$

It is interesting to notice here that this increase in the objective function could also be obtained if we increase the supply of the capacity b with

$$\frac{\Delta a_{bf} \cdot \bar{p}_f}{1+y_{fb} \cdot \Delta a_{bf}}$$

This point is of great interest to management in the following respect. Suppose it must be decided whether to increase the capacity of a given facility or to try to make a more efficient utilization of the existing facility. Then, if an increase of

$\Delta p_b$ will cost $k_1$ and the decrease in $-\Delta a_{bj}$ may cost $k_2$, where $k_1 < k_2$, then management should reduce $a_{bj}$ by $\Delta a_{bj}$ since both courses of action will bring about exactly the same incremental benefits. (1)

5.6.4 Addition of a new variable.

To examine the possibility of introducing the new variable $x_{n+1}$, we must determine its profitability chance through:

$$\beta_{n+1} = \beta_r \cdot a_{r(n+1)} - c_{n+1}$$

if $\beta_{n+1} < 0$ then the introduction of $x_{n+1}$ will be profitable, otherwise it will be unprofitable if $\beta_{n+1} > 0$, or neutral if $\beta_{n+1} = 0$.

If $\beta_{n+1} \leq 0$, further iteration(s) must be accomplished.

5.6.5 Addition of a new constraint.

Suppose a new constraint is to be added to a LP problem for which an optimum has already been obtained. Let $b_{m+1}$ be the new constraint then

$$\bar{p}_{m+1} = p_{m+1} - a_{m+1,r} \bar{p}_r$$

if $\bar{p}_{m+1} \geq 0$ the same solution will be optimal and feasible. On the other hand if $\bar{p}_{m+1} < 0$, the solution will be no longer feasible. In this case we must define $y_{m+1,r}$ through

$$\sum_{r=1}^{k} y_{m+1,r} = - a_{m+1,r} \cdot y_{rr}$$

(1) IBID , p. 315 .

and $y_{m+1,s}$ · s = k+1, k+2,...,n, through:

$$y_{m+1,s} = y_{m+1,r} \cdot A_{rs} - a_{(m+1)s}$$

Having defined the vector $y_{m+1,j}$, further iteration(s) must be performed untill a feasible solution is obtained.

Defining the whole $\sum_{j=1}^{n} y_{m+1,j}$ would enable us to determine which negative element ( ) $y_{m+1,j}$, must be selected as a pivot element in order to maintain the feasibility or the optimality conditions; or both of them. It is suggested, however, to assure, first, a feasible solution.

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( ) Since $\bar{p}_{m+1}$ 0 the pivot element must be negative in order to have a feasible solution.

# 6. The modified KAIRO METHOD : Evaluated.

## 6.1. How many operations per iteration are required?

In this chapter, we will try to make a rough estimation of the total number of computations required to perform:

First: K iterations, assuming that the basis increases from one iteration to another, which means that at the Kth iteration, the basis is of K order.

Second: One iteration assuming that at the Kth iteration there were only R real variables in the basis.

Having defined the inverse of the current basis $A_{rr}^{-1}$, the relative cost coefficients $\beta_r$, and the value of the constant vector $P^o$, to move towards an adjacent extreme point, the following steps must be performed.

1 - Defining the relative cost coefficients for the nonbasic variables $\beta_s$

2 - Determining, $Y_e$, the value of the newcoming variable $x_e$ in terms of the current basis

3 - Determining the pivot row

4 - Determining the value of the outgoing row in terms of the current basis

5 - Updating the new basis

6 - Determining the relative cost coefficients of the real basic variables $\beta_r$

7 - Determining the current value of the constant vector $P^o$

Operations required for step 1. (§)

At the first iteration, no operations are required.

At the second iteration (n-1) operations are required.

At the third iteration 2(n-2) operations are required.

... ... ...

At the Kth. iteration k-1(n-k+1)

Total = $1/2\ k^2n - 1/3\ k^3 + 1/2\ k^2 - 1/2\ kn - 1/6\ k$

Proof: Let us assume that we perform k+1 iterations. The total number of operations is

$= 0 + 1(n-1) + 2(n-2) + \ldots + k(n-k)$

$= 0+m-1+2n-4+ \ldots + kn-k^2$

$= 0 + (n+2n+\ldots+kn) - (1+4+9+\ldots+k^2)$

$= k/2\ (n+kn) - (1/6k\ (k+1)\ (2k+1))$

L $= 1/2kn + 1/2\ k^2n - 1/3\ k^3 - 1/2\ k^2 - 1/6\ k$

Since the (k+1) th. requires k(n-k), then the operations required only for k iterations

$= L - kn + k^2$

$= 1/2\ k^2n - 1/3\ k^3 + 1/2\ k^2 - 1/2\ kn - 1/6\ k \ldots I$

Operations required for step 2.

Evaluating the newcoming variable in terms of the current basis.

At the first iteration, no operations are required.
At the second iteration, m operations are required.
At the third iteration, 2m operations are required.
... ... ...

§) When discussing the number of arithmetic operations involved in any computation, we will only count multiplications and divisions

At the kth. iteration, k-1(m) operations are required.

Total = O + m + 2m + ... + km -m

= 1/2 k (O + km-m)

= 1/2 $k^2$m - 1/2 km .....II

Operations required for step 3.

Finding the min nonnegative of $p'^{o}_{i}$ / $a'_{ie}$

km .....III

Operations required for step 4.

Evaluationg the outgoing row in terms of the current basis.

At the first iteration, no operations are required.
At the second iteration, one operation is required.
At the third iteration, 4 operations are required.
At the fourth iteration, 9 operations are required.
... ... ...
At the Kth. iteration, $(k-1)^2$ operations are required.

Total = O + (1 + 4 + 9 + ...+ $(k-1)^2$)

= 1/6 (k-1) (k+1-1) (2(k-1)+1)

= 1/3 $k^3$ - 1/2 $k^2$ + 1/6 k.....IV

Operations required for step 5.

Updating the current basis.

At the first iteration, one operation is required.
At the second iteration, 4 operations are required.
At the third iteration, 9 operations are required.
.... .... ....
At the K+h. iteration, $k^2$ operations are required.

Total = 1/6 (k+1) (2k+1)

$= 1/3\ k^3 + 1/2\ k^2 + 1/6\ k$ .....V

Operations required for step 6.

Transforming the constant column vector $p'^{o}_{i}$

km .....VI

Operations required for step 7.

Transforming the relative cost coefficients -row vector- $\beta_r$ .

At the first iteration, one operation is required.
At the second iteration, 2 operations are required.
... ... ...
At the Kth. iteration, k operations are required.

Total = 1/2 k (1+k)

$= 1/2\ k^2 + 1/2\ k$ .....VII

Summing up the total operations required for K iterations:

$1/2\ k^2 n - 1/3\ k^3 + 1/2\ k^2 - 1/2\ k\ n - 1/6\ k$ .....(I )

$+ 1/2\ k^2 m - 1/2\ km$ .....(II )

$+ km$ .....(III)

$+ 1/3\ k^3 - 1/2\ k^2 + 1/6\ k$ .....(IV )

$+ 1/3\ k^3 + 1/2\ k^2 + 1/6\ k$ .....(V )

$+ km$ .....(VI )

$+ 1/2\ k^2 + 1/2\ k$ .....(VII)

$$K^2(1/2\ n + 1/2\ m + 1/3\ K + 1) + 2/3\ K + 1/2\ K(3\ m - n) \quad (\ )$$

if n = 3 m, the previous estimation would be:

$$K^2(2\ m + 1/3\ K + 1) + 2/3\ K$$

Considering the number of operations required to perform one iteration applying the RSM

$$3\ m^2 + m\ n \qquad (1) \qquad \text{operations}$$

if n = 3 m we may require [illegible] operations which equal to the number of the operations required by the standard simplex method if [illegible].

for K iterations we need:

$$6\ K\ m^2 \qquad \text{operations}$$

a quick substitution for K,m and n under any condition, would show how superior the MKM over the RSM is.

Even for smaller n, the MKM shows superiority on the standard simplex method. (2) For [illegible], the MKM would require [illegible] operations per iteration. For K iterations the SM would need [illegible]. If [illegible] the operations needed turn to be $2m^3$ which is much more than the [illegible] required by the modified Main method.

---

1) Wagner, H. M., op. cit. pp 361-369

2) The other advantages of the MKM over both of the simplex and the revised simplex methods will be discussed soon.

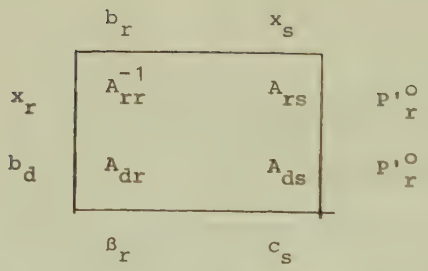
## 6.2. Comparison of the SM, RSM and the MKM.

The previous presentation is not able to reflex the real superiority of the MKM over the SM and the RSM. The number of the calculation made for the MKM was based on the assumption that at every iteration a new non-basic variable will be introduced and a new constraint will be satisfied. This means that the number of the basic variables, being in the solution, is always the same as the number of iterations performed. Practically, such an assumption is not always true. During the cours of the iterations it happens that either one of the basic real variables may turns to be nonbasic or one of the already satisfied constraints may assume a non-zero level. Both cases, however, may occur simultaneously. Such a fact would lead us to conclude that the estimated number of operations made for the MKM is a misleading one. Because when one of the basic real variables leaves the basis or one of the satisfied constraints assumes a non-zero level, a considerable amount of calculations is saved. It happens that the basis for many successive iterations does not increase at all if not decrease. (§)
The following example may explain this point:

Let the "simplex" tableau at the beginning of the (k+1) st. iteration as follows:

---

§) see example solved pp. 160-172




Performing the (K+1)st iteration, we may face one of the following cases:

case 1: when the pivot element is an element of $A_{ds}$

case 2: when the pivot element is an element of $A_{rs}$

case 3: when the pivot element is an element of $A_{dr}$

case 4: when the pivot element is an element of $A_{rr}^{-1}$

Recalling the different 7 steps required to perform one iteration and assuming that the number of the real variables introduced in the basis is R, we may need:

| | | |
|---|---|---|
| step 1: | $(n-R)R = n\ R - R^2$ | (1) |
| step 2: | $mR$ | (2) |
| step 3: | $m$ | (3) |
| step 4: | $R^2$ | (4) |
| step 5: | $(R+1)^2 = R^2 + 2R + 1$ (§) | (5) |

---

§) Notice that the basis now is of (R+1) order

step 6: $R + 1$ (6)

step 7: $m$ (7)

Summing up 1 to 7 we have

$$R^2 + R(n + m + 3) + 2m + 2$$

On the other hand, if we have to deal with case 2, we may save:

First: $R^2$ operations required to define the outgoing row, since the outgoing variable is already being defined in $A_{rr}^{-1}$

Second: $(R+1)^2 - R^2 = 2R + 1$

Since we will still dealing with R dimension

Third: $(R+1) - R = 1$

since the dimension of $\beta_r$ will not increase

Hence, we may need only

$$R(n + m + 1) + 2m$$

to perform the (k+1)st iteration.

In case 3 we may save:

First: $R^2$ operations required to evaluate the newcoming variable, since its value is already being defined in $A_{rr}^{-1}$

Second: $2R + 1$ (see case 2)

Third: 1 (see case 2)

as in case 2, to deal with case 3, we may only need:

$$R(n + m + 1) + 2m$$

In case 4 we may save

First: $2\ R^2$ operation required to evaluate both of the outgoing row and the newcoming variable in terms the current basis

Second: $(R+1)^2 - (R-1)^2 = 4\ R$ operations
since the dimensions of the basis will be of (R-1) order

Third: $(R+1) - R-1) = 2$ operations
since the dimensions of $B_r$ will decrease.

In such a case we may need

$$R(n+m) + 2\ m - R^2 - R$$

operations to perform the (K+1)st iteration.

Here, once more, it might be clear that under any conditions the MKM is better than the RSM taking in our consideration that the MKM enjoys -at least- all the advantages of the RSM. This point will be discussed soon.

Regarding the standard simplex method, we may notice that in cases 2,3,and 4 the MKM is absolutely better than the SM. In case 1 the MKM is better even when $n = 3/2\ m$ and R is equal to m. If $n = m$, the SM begins to be better [illegible] $R \geq 2/3\ m$.

In such a case the saving which would have been achieved during the course if iterations must not be neglected. It might be as much as:

$$\frac{4}{9}\ m^3 \qquad (*)$$

---

*) We assumed that n=m and K=R=2/3 m and substituted in ( ) -for the MKM. For the SM we substituted in R(2 mn). is assumed, as well, [illegible] ...ore of Case 1.

Perhaps it might be convenient here to make a quick estimation to the number of the computations required to perform an iteration when the product form of the inverse is applied.
Let us assume that at the end of the kth iteration, the number of the vectors E is R. To move towards an adjacent extreme point, the following steps must be performed:
Step 1. Calculate the current relative cost coefficients applying the formula:(1)

$$\beta_j = C_B B^{-1} A_j - C_j$$

by using the form:

$$\beta_j = F A_j - C_j$$

where $F = C_B B^{-1}$

the row vector F must be first computed by the recursive formula:

$$F = (((C_B E_R) E_{R-1}) \dots E_1)$$

To perform this step we first need mR operations to compute F and R(n+m-R) to compute the relative cost coefficients of the nonbasic variables, hence we would require:

$$mR + R(n+m-R)$$

Step 2. Determine $Y_{ie}$ which is the value of the new-coming variable $x_e$ in terms of the current basic solution. This is done applying the formula:

$$Y_{ie} = (E_R (E_{R-1} \dots (E_1 a_{ie})))$$

here we would need

mR operations

---

(1) Luenberger, D., op cit p. 59

Step 3. Define the value of the constant vector $\bar{P}_i^o$ . In the literature, this step is accomplished using the recursive formula:(1)

$$x_B = \bar{P}_i^o = ( E_R ( E_{R-1} \ \ldots ( E_1 \ P_i^o \ ) ))$$

this would require $mR$ operations.

It is interesting to indicate that this step is superflous . The value of the vector $\bar{P}_i^o$ could be directly defined, as applied by the MKM (see p. 130), using the usual transformation procedure:

$$\bar{P}_g^o = P_g^o / y_{ge}$$

$$\bar{P}_i^o = P_i^o - Y_{ie} \cdot P_g^o \qquad \begin{matrix} i=1,2,\ldots,m \\ i \neq g \end{matrix}$$

doing that we would require only $2m$, since the vector $Y_{ie}$ is already being defined at the second step .

Finally we would need another $m$ operations to define $E$ using the formula:

$$V_i = - Y_{ie} / \ y_{ge} \qquad i=1,2,\ldots,m \ \ i \neq g$$

$$V_g = \ 1 / \ y_{ge}$$

Summing up the required operations we would require:

$$3mR + Rn + 3m - R^2$$

Recalling that the modified kairo method requires at worse $R^2 + R(n+m+3)+2m+2$ (p. 148) , it would be interesting to notice that the MKM shows superiority over the product form of the inverse until $R = m-1$ where both of the two methods require exactly the same amount of operations. However, we may argue that the MKM is absolutely better than the product form of the inverse for the following reasons:

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(1) See Orchard-Hays, W., op cit P. 73, and Simmonard, M., op cit p. 57

1- The number of operations required by the MKM is a maximum estimation, since the other three cases (see p. 145 ff and p. 197 f) may occur where much less operations are required, while the estimation made for the product form of the inverse is an optimistic one since at the Kth iteration there could be K vectors where K > R. We would have R vectors only when performing the reinversion procedure. (1)

2- The MKM eliminates automatically the irrelevent vectors and gives at every iteration a minimum amount of information in a compact form.(§)

3- The MKM generates at every iteration the inverse of the current basis and no extra computations are required to define it.(§§)

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(1) Garvin-Hays, W., op cit p. 73
Orchard, M., op cit p. 87
Graves, L., and Wolfe, P., op cit p. 213

(§) Examine the example solved (pp. 160-172) to see how this process is done

(§§) The other advantages of the MKM are discussed on pages 206-207

## 6.3. Further advantages of both of the KM and the MKM:

We have seen that the new methods permit us to travel "light" over the surface of the convex set having no excess baggage of the type $(-A_{dr} \cdot A_{rr}^{-1})$.

Both of the KAIRO METHOD and its modified form give a new insight to the inverse of the basis and generates it numerically and facilitate post-optimization studies and give a new interpretation to the change in the input-output coefficient $a_{ij}$.

Applying the KAIRO METHODS, the numerical information is carried along in much more compact form. The decision maker is releaved from examining irrelevant information when dealing with a basis of R dimensions instead of m.

Another advantage of KAIRO METHODS is that fewer entries need to be made in each iteration. This aspect is very important when dealing with large scale problem. For the simplex method, essentially (m+1)(n+1) elements must be written in each tableau.

For the revised method (m+1)(m+2) elements need to be recorded. For product form of the inverse a column vector and its associated index. With the KM, since the inverse is generated at each iteration, we need only to record the index of the variables introduced and the constraints exhausted. Applying the MKM, we need only to record a small basis of R dimensions and we may avoid the tiresome job of evaluating the required information, when the product form of inverse is used.

Furthermore, the KAIRO METHODS make benefit of the existence of the high percent of zero elements since the zero elements are not subject to change until it enter the basis. Moreover, they enable us for the first time to solve a LP problem without using slack variables.

A final advantage of the KAIRO METHODS lies in the control of rounding-off errors. Applying the KM, the inverse is computed directly at each iteration by means of the original $a_{ij}$. Both of the KM and the modified form allows rounding-off errors to be controlled in another way. The elements $Y_{dr}$ in the simplex method and its variants are subject to multiplication and division operations at each iteration where every multiplication or division increases the rounding-off errors.
Applying the KM and its modified form, the elements $Y_{dr}$ are not generated until one of the $A_{dr}$, namely $a_{gr}$, is chosen to be the outgoing row.

This reduction in the number of operations implies a corresponding reduction in the rounding-off errors.

Another point ist that, applying the new methods, there is an assurance that the dimensions of the basis R, will never exceed the number of the constraints or the number of the variables, as might happen when the product form of the inverse is used. Such a fact which necessitates the frequent re-inversion.
An important disadvantage of the KAIRO METHODS is the fact that the pivot element must be choosed according to the steepest ascent rule. Trying to define the variable which will give the greatest increase in the objective function will be too much expensive. Such a rule, however, does not prove to be always effective.

At last, we must stress the fact that both of the KAIRO METHODS are not suitable at all for hand computation. But since real problem are solved only on digital computers, the above disadvantage is of no importance.
KAIRO METHODS have been devised to more efficiently utilize the digital computers.

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# Lebenslauf

Am 12. September 1940 wurde ich als erster Sohn des Kaufmanes Mohamed Gouda und seine Frau Fatma in Tala/Ägypten geboren. In Tala besuchte ich bis 1954 die Primarschule. Da es keine Sekundarschule in Tala damals gab mußte ich in Tanta, einergrößeren Stadt, lernen. Juli 1958 bestand ich mein Abitur.

Vom Oktober 1958 bis Juli 1962, habe ich acht Semester BWL studiert und mit dem Ba chelor of Commerce an der Universität Kairo abgeschlossen.

Vom Januar 1963 bis November 1966 habe ich als Assistent im Nationalen Institut für Höhere Verwaltung in Kairo gearbeitet. Während dieser Zeit machte ich ein Vier-Semesteriges Diplom in BWL (1963-1965), ebenfalls an der Unversität Kairo. Vom November 1964 bis Juli 1965 nahm ich am " Ausbildungsprogramm für Verwaltungsentwicklungsexperten" am Nationalen Institut für Höhere Verwaltung teil.

Vom November 1966 bis September 1970 habe ich am Institut für Handelslehre in Lybien gearbeitet. Während dieser Zeit übersetzte ich fünf Bücher vom Englischen ins Arabische , die mit Erfolg verlegt worden sind. Aufgrund dieser Tätigkeit bin ich vom Sebtember 1970 bis November 1971 in der Redaktion der El-Tawrah Zeitung in Tripolis tätig gewesen, wo ich verschiedene betriebswirtschaftliche Artikel geschrieben habe.

Im Dezember 1971 bin ich in die Bundesrepublik Deutschland gekommen. Bis Ende Januar 1972 habe ich die deutsche Sprache im Göethe Institut in Staufen gelernt.

Seit dem Sommer Semester 1972, bin ich an der Universität Göttingen eingeschrieben und promoviere bei Herrn Professor Dr. J. Bloech.